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Physics. — DEBIJE–SCHERRER *exposures of liquid helium.* By W. H. KEESOM and K. W. TACONIS. (Abstract of Communication N⁰. 252*c* from the KAMERLINGH ONNES Laboratory at Leiden.)

(Communicated at the meeting of February 26, 1938.)

DEBIJE–SCHERRER exposures on a jet of liquid helium were made.

We found for helium I at 2.5° K as well as for helium II at 1.6° K a liquid diffraction ring with an angle of deviation of 28° (Cu K α rays).

For liquid helium I the diffraction ring corresponds to the formula one of us and DE SMEDT found valid for substances with simple spherical molecules.

For liquid helium II we discussed the diamond lattice hypothesis advanced by F. LONDON and used by FRÖHLICH to explain the behaviour of the specific heat. We did not find this hypothesis in harmony with the results of our X-ray examination. We found, however, another lattice that may serve in a theory as F. LONDON's as well, that probably still better fulfills the requirements connected with a theory such as FRÖHLICH's, and that, moreover, agrees with the X-ray result. This structure, which can be derived from the face-centered cubic lattice by omitting half the number of atoms, belongs (if the atoms are taken as fixed) to a space-group T_d^2, the elementary cell containing 16 atoms with co-ordinates $\frac{1}{8}$, $\frac{1}{8}$, $\frac{1}{8}$. In this structure each atom has six neighbours. We mention further that the holes (places of omitted atoms) are situated in rows, so that they form a sort of channels. These channels may possibly lead to an explanation of the very great thermal conductivity of helium II.

Geology. — *Lateral movements on the Alpine Foreland of Northwestern Europe*. By W. A. J. M. VAN WATERSCHOOT VAN DER GRACHT.

(Communicated at the meeting of February 26, 1938.)

It becomes ever more apparent that the major inter-continental orogenies of the world must be considered as the result of *unilateral* push, caused by one major continental unit of the crust encroaching upon another, in the sense of ARGAND, F. E. SUESS and others. Large continental blocks must be conceived as being laterally displaced, notably during certain worldwide periods of crustal unrest. The result is usually that the edge of one block is overridden (but occasionally also underthrust) by the edge of the encroaching unit, with duplication, or at least considerable thickening of the crystalline crust. This blocks the outflow of internal heat, increasing the geotherms, and simultaneously causes intense crushing of the inter-continental sedimentary mantle (the geosyncline) beneath the overriding masses. Minor backfolding is frequently in evidence at the rear of the orogenetic structure (Alpine Dinarides, Variscan back-folding and slicing in the southern Schwarzwald and the Vosges Mountains and in southern Moldanubia).

The character of the resultant more or less intense regional metamorphism of the rocks involved may serve as a help to unravel the usually very complicated mechanism. Kata-metamorphism is generally confined to the crystalline basement (not necessarily pre-Cambrian) of the blocks involved; it is not a result of the tectogenetic diastrophism, although in the enorogenetic zone kinetic metamorphism may have reworked the physico-chemical structure of the rocks entirely.

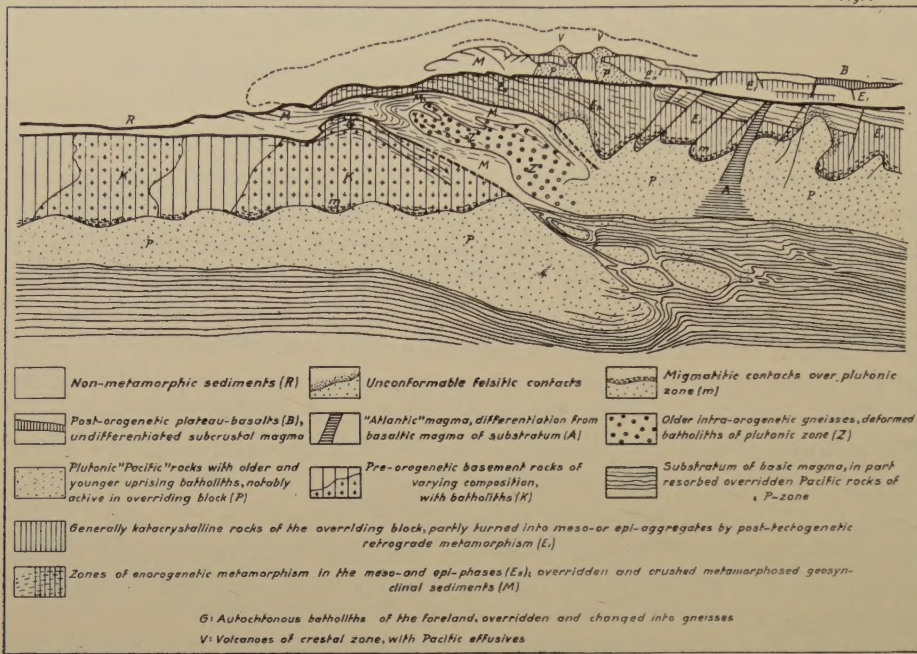
On the overriding elevated block erosion has frequently deeply bared these kata-metamorphic horizons; they occasionally also appear on elevated massifs piercing the sedimentary mantle of the autochtonous foreland, or in erosional windows of the frontal thrustsheets. The intra-orogenetic zone of intensily deformed sediments of the geosynclinal substratum under the overriding block is usually in the meso- or epi-metamorphic phase, but may occasionally contain kata-metamorphic rocks of the substructural basement, which may or may not have been subjected to retrograde metamorphism and turned into meso- or even epi-crystalline aggregates. Such poly-metamorphism frequently complicates the picture **(11)**.

The major crustal overthrust (basement thrustplane — "charriage de fond" of ARGAND) evidently descends into unexpectedly considerable depths, as is proven by deep-focus earthquakes of the intermediate class (depths from 70 to 250 km), the epicenters of which invariably fall on or close to the major tectonic lines of Tertiary or more recent orogenies,

in parallelism to the lines of epicenters of normal high-focus shocks, the deep intermediate foci being farther back of the mountain front than the

Schematic Section of an inter-continental orogenetic zone

Fig. 1



Mostly after F.E. Suess.

Fig. 1.

shallow shocks. The deep-focus earthquakes, as far as we know (meaning shocks at depths of 300—650, but not exceeding 700 km), are confined to the circum-Pacific ring and must have a different cause, scarcely connected with normal orogenetic stresses. Anyhow, therefor, the normal crustal disturbance of major orogenies apparently descends to depths of at least up to 250 km (East-Indies, Caribbean, Andes) (6). Smit Sibinga even asserts that in the Netherlands East Indies, therefor within the Pacific ring, even the deep foci remarkably coincide with the major orogenetic lines, as indicated by the gravitational work of Vening Meinesz and morphologic discontinuities. Consequently, in the circum-Pacific region, orogenetic structure may even descend into depths of up to 600 km. (8)

That such intense interaction of drifting crustal masses of continental magnitude cannot have affected only the boundary zones of the colliding blocks, causing the major inter-continental mountain belts, is evident. The interior of the continents (the foreland) has also become warped, dislocated, and contains minor, but yet often important, intra-continental orogenies, but regional enorogenetic metamorphism has rarely been caused by such movements. It is admitted that it is not easy to explain exactly

in which manner these stresses have been transmitted over long distances in an always more or less labile crust. Only it is evident from a concurrence of numerous observational facts, that pressure was active a long distance in front of the orogenetic belt, either by direct transmission through the actual crust, or by indirect action by way of still more labile subcrustal masses, dragging the super-structure like a current carrying floating ice.

The continental blocks are far from homogeneous. They contain units of greater or lesser rigidity, positive basement or substructural massifs next to negative regions (shelves), subject to repeated depression into basins and sedimentary troughs. These latter are more easily subject to deformation through lateral compression between more rigid massifs. In part this multifareous behavior may be caused by batholithic masses in the deep basement, or it may be caused by older orogenetic structures of a former phase, now solidly incorporated in a continental crustal unit, but having retained a superior individual coherence and rigidity (substructural massifs). In a former treatise the writer has attempted to describe the structure of the Midcontinent Plateau of the North-American Continent, as caused by stresses originating from the major late-Paleozoic orogeny (Appalachian-Ouachita Mountains) along its outer border, an inter-continental branch of the worldwide Altaids (**12**). The present paper intends to analyse briefly some of the post-Paleozoic deformation of the continental mass of northwestern Europe, especially from the viewpoint of pressures affecting the continental block from the south, culminating into the Alpine orogeny on its southern rim, against the resistence in the north from the Fennoscandian nucleus. Next to this vise-action an independent westward tension is to be taken into account, originating from the opening Atlantic in the sense of WEGENER.

The European continental block shows a far more complicated mosaic of incongruous units than North-America. These have become compressed and jostled between the southern edge of the primeval Fennoscandic Shield and the northward pushing masses in front of the successive Variscan and Alpine diastrophisms. These two latest orogenetic cycles, however, and the pressures to which they subjected their respective forelands, are not the only cause of the deformation of the Northwest-European continent. All of northwestern Europe is traversed by a widespread major system of great WNW to NW trending lines of dislocations (lines of KARPINSKY), paralleling the southwestern rim of the Russian Plateau (from Galicia, Bromberg, Bornholm to Skone). These dislocations, sometimes overthrust, are in evidence from the Black Sea to Eastern England and independently cross the Variscan arc. The resistence of the old Russian-Fennoscandian nucleus to the Variscan and Alpine pressures from the south may be in a part the cause, the feature however, is of such magnitude that an underlying superior cause may be the explanation. The continent is also subject to still another, evidently

entirely independent strain, in no way connected with the interaction between the European and Gondwana continental blocks, causing major N-S directed dislocations, originating particularly in the middle-Tertiary, although older similarly directed lines of weakness are indicated. These express themselves as *rifts* of a similar nature as those which disrupt the African continent; they must be regarded as a result of stretching by the same major crustal drifting that opened the North-Atlantic.

The Alpides entirely overwhelm the older Variscan arc in the east of Europe, approaching the Russian Plateau along the Dniester river in Poland. Farther to the west the Alpine foreland embraces an ever widening expanse of territory, which had previously been subjected to the Variscan revolution. In southwestern Europe the foreland is divided by the intra-continental Pyrrenean orogeny. The also Variscan Paleozoic substructure of Iberia, south of the Pyrrenees, is left out of the scope of the present paper.

The morphology of the Alpine foreland, as we know it at present, is a relatively recent development; the present topography is even of extremely late origin. The entire area is generally characterized by more or less rigid tilted blocks, sliding over or under and past each other, under stresses acting on them from the south, deforming the sedimentary filling within intervening downwarped basins, whilst the elevated massifs often show the Paleozoic substructure on their surface bared by erosion. These jostling "epirophoretic" movements, comparable to those in drifting ice-floes, originated in the middle-Mesozoic (Jurassic), they culminated in the late-Cretaceous and had another active phase in the middle-Tertiary, gradually subsiding but still noticeable in the Pleistocene and probably not yet extinct. The general lines of deformation are clearly outlined on the map of figure 2, after von SEIDLITZ (7). The relatively more rigid exposed massifs, remnants of the Variscan basement, are shaded.

Many of these dislocations and fractures, which indiscriminately disrupt even such old kata-metamorphic massifs as those of the Moldanubian zone of the Variscan orogeny, are revivals of the older Paleozoic, or even a more ancient pattern. As instances might be mentioned the repeatedly reactivated great quartz lodes of the Bohemian and Bavarian "Pfahl"s and the overthrust of the Lausitz granite massif over upper-Cretaceous. The "Frankische Brüche" bordering the Bohemian massif on the west, and the great thrustfaults on its southern rim over the Basin of the Danube are similar instances. In the interior of the Bohemian massif the faults between Eger and Bodenbach, and some others, were sufficiently deep fractures that they opened the way for sub-crustal basic basalts; they also cause a number of famous hot springs (Karlsbad and many others). The many transverse faults of the Variscan arcs were reactivated

in the Jurassic and again in the late-Cretaceous; locally the movements restarted in the later-Tertiary and even in the Pleistocene. The basins

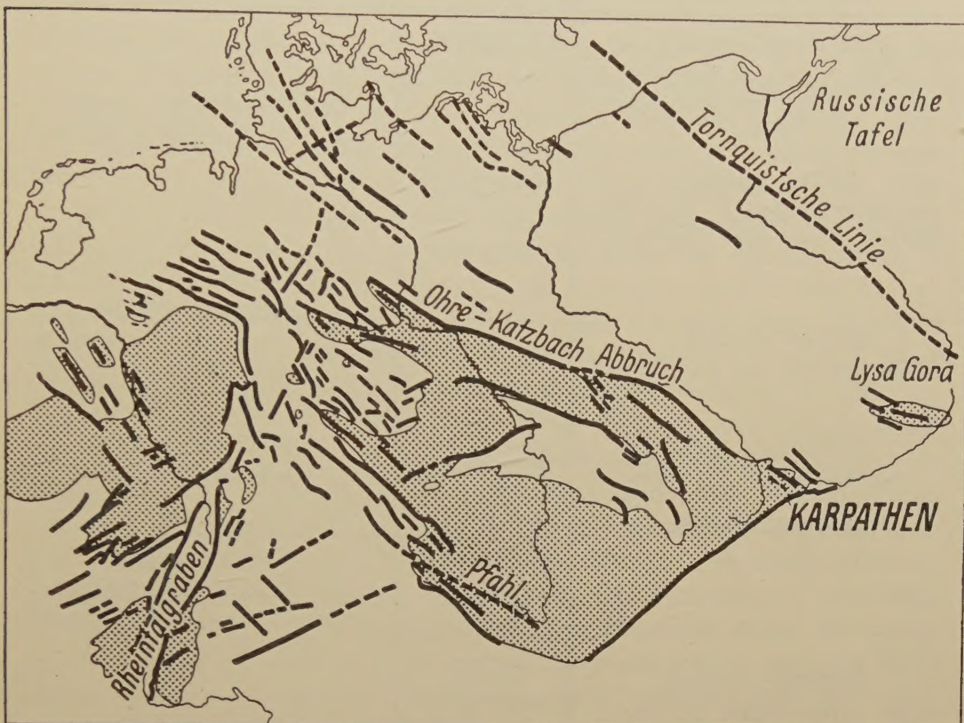


Fig. 2. Lines of Mesozoic (Saxonic) deformation. After V. SEIDLITZ, e. al.

between the positive blocks originated or were deepened also late in the Jurassic; they were compressed and folded principally in late-Cretaceous time. The salt domes of the deep basin of the northwestern plains show the same phases in the uprising of their saltplugs, proving that the same stresses continued to act on the subsurface that far to the north.

All these "Saxonic" movements show a distinct relation to the Alpine phases in the south, but they were not strictly contemporaneous. The following comparative Table shows them in juxtaposition. It is interesting to note that important warping and fracturing occurred on the foreland (Kimmeric phases) before notable deformation became apparent in the Alpine mountain belt. Presumably this was the time that deep in the crust the major inter-continental basement overthrust came into existence and placed the European foreland under considerable pressure, before serious deformation reached the surface in the actual Alps.

The scope of this paper does not permit to go into much further detail concerning the interesting movements on the many separate units which have been studied and described in the literature (4). It may suffice to review briefly the late-Mesozoic-Tertiary history of one of the major

OROGENIC PHASES IN ALPINE CHAINS	FOR-MATIONS	SAXONIC OROGENIC PHASES IN FORELAND
RHODANIC PHASE: second folding of Jura Mts, Molasse Basin and Sub-Alpine chains.	LEVANTIC	important regional changes of level on foreland blocks.
ATTIC PHASE: first major folding of Jura Mts. Molasse Basin (and Caucasus).	PONTIC	
	SARMATIC	
STYRIC PHASE: in Eastern Alps and Carpathians.	HELVETIC	MIOCENE PHASES: in numerous Basin-structures in western Germany and SE-England.
PYRRENEAN PHASE: major folding of Pyrrenees.	LOWER-TERTIARY	(Continental rifting, chiefly in Oligocene).
LAMARIC PHASE.	SENONIAN	LAMARIC PHASE: reversion of creep. SUB-HERCYNIC PHASES:
	TURONIAN	northward creep of basement.
AUSTRIC PHASE: major initial phase in Alpide chains of Europe: Alps, Caucasus, Carpathians, Dinarides.	CENO-MANIAN	marine transgressions.
	LOWER-CRETACEOUS	UPPER-KIMMERIC PHASE: major initial phase of Basin structures.
faint deformations in Alps.	JURASSIC	OLDER-KIMMERIC PHASE.

substructural units of Central Europe, the Rhenisch-Ardennic block (Rheinische Schiefergebirge and Ardennes), inclusive of the Hunsrück-Taunus, which belong to the frontal Variscan zone, and its interaction with the pre-Variscan foreland-block, the Brabant Massif, in the northwest, and the Central-German block (Mitteldeutsche Hauptscholle) in the east. The latter is a much more broken and differentiated unit, comprising the Erzgebirge in the southeast, and the spurs of the Thüringer Wald, the Harz and the Flechtinger Höhenzug, with its continuation in the Lusatische Schwelle in the northeast. Its northern half is broken by two deep labile basins: the Subhercynic Basin, between the Harz and the Flechtinger Zug, and the Basin of Thüringen, between Harz and Thüringer Wald, both with numerous complex interior structures. Present morphology is exclusively a result of Mesozoic-Tertiary deformation, which is practically independent from the Paleozoic Variscan structure. The tectonic lines of the map of figure 2 strongly suggest the general

creep towards the north, complicated though it may be by the crossing NNE zone of west-European rifting.

The present outline of the Rhenish-Ardennes block dates from the old-Kimmeric phase, in the Lias, and to a greater extent than in the eastern block its form is controlled by tectonic lines, dating from the Variscan orogeny. The southern boundary coincides with the late-Variscan Saar-Selke trough, but the present scarp is a result of very late-Pliocene and Pleistocene uplift of the Hunsrück-Taunus ridge (250 m uplift in Plio-Pleistocene!). The deep-faulted Embayment of Cologne (Kölnische Bucht) and the related Neuwied Basin are a rejuvenation of an older line of weakness around the southeastern spur of the Brabant Massif, which expressed itself already in the Variscan structure (Eifel sigmoid) and again in Triassic time (Nassau straits). The N-S trend of the Rhine rift intermingles here with NW-crossfaults of Variscan origin, rejuvenated in Kimmeric time. The northern and eastern edges of the substructural Rhenisch block (Rheinische Schiefergebirge) are Kimmeric. In the Jurassic the already preexistent swell (the Zechstein sea never covered this block) differentiated more sharply from the now rapidly deepening Basin of Hannover. The Mesozoic shoreline (Niedersächsische Uferrand) became more pronounced. The Hessian depression deepened. Late in the Jurassic, by the upper-Kimmeric phase, the northward pointing rim of the massif became surrounded by a chain of faultfolds: the pre-Cretaceous Egge range, which extended into the, afterwards differentiated, Lippische Schwelle a little farther north (**9**). The Paleozoic substructure became involved into this folding and was uplifted all along the northeastern boundary of the original block, with detached frontal uplifts at Detmold and notably around Osnabrück; at the latter locality even now the Coalmeasures are exposed. West of Ibbenbühren the structures of this phase become much less distinct under the rapidly increasing cover of more recent sediments, but apparently the Kimmeric deformation now spreads considerably farther to the south. Around Winterswyk, in the Netherlands, similar faultblocks as we find in the Egge are elevated and bring Lias, Trias, Zechstein and even Coalmeasures close to the surface. Here the movements are in part old-Kimmeric, as well as upper-Kimmeric. All through Twenthe we know similar, although not quite as highly uplifted blocks bordered by NW-striking faults. Positive gravitational anomalies indicate the continuation of this NW to WNW striking zone of highs in the deeper subsurface all through the Province of Overyssel, as far as the shore of the Zuiderzee and southwestern Friesland (with a deep basin to the northeast).

The next episode was a very different one. Tectogenetic pressure temporarily comes to rest but, at the time of the widespread transgression of the upper-Cretaceous all over western Europe, a very deep upper-Cretaceous basin (Münsterbecken) develops over the nothern half of the Rhenish Massif, considerably deepening the already dishlike depres-

sion of the Paleozoic surface by the uplift of its northern boundary in the Egge chains and the Winterswyk-Twenthe blocks. Between Münster and Koesfeld the thickness of the upper-Cretaceous cover (complete from Cenomanian to upper-Senonian) reaches over 1500 m. No upper-Cretaceous at all was deposited over the eastern Netherlands as far north as Ootmarsum (Erkelenz-Swell). The Cretaceous cover entirely buried the already baseleveled Egge structures.

It was only during the Senonian, beginning in Emscher time, that the tectogenetic pressures were renewed and now became particularly active. A differentiated northward drift of the Paleozoic substructure now becomes particularly conspicuous. It was at this time that the Harz block and other ridges of the eastern "Hauptscholle" were uplifted, tilted to the south, and their elevated northern and northeastern edges overthrust towards the north. This is particularly well demonstrated for the northern edge of the Harz block (2000 m throw). The original gentle swell of the Harz carried a cover of Zechstein, Trias, Jura, Cenomanian and Turonian; it seems that the block did not even form a shoal in the Zechstein sea. An old-Tertiary (presumably Eocene) peneplain lies now at + 1000 m above ordnance datum. The Brocken granite stood out above it as a low hill of some 120 m. Off the northeastern rim of the block the Paleozoic platform lies now at — 2000 m, indicating an aggregate uplift, at least difference of level (exclusive of erosion on the Harz block itself), of 3000 m. In the area of Hannover the Paleozoic platform should lie at — 3000 to — 3500 m, a difference of an additional 1500 metres. The uplift of the Flechtinger Höhenzug was less important: the difference in level between the Paleozoic outcrops and the platform off the NE-rim is around 1300 m.

On the Rheinische Masse the northern edge of the substructural block, with its adhering boundary zone of the Kimmeric Egge structures, was thrust bodily under the newly uplifted chain of the Osning (Osning Ueberschiebung), over a distance of at least 2 km (as proven by erosional windows). (9)

In the Hessian depression, the now thoroughly labile Basin of Thüringen, and the Subhercynic Basin south of the Flechtinger Höhenzug, a number of structures were uplifted, greatly complicated by the plasticity of intercalated layers of rocksalt. The overthrust of the Osning may also be explained as an instance of salt tectonics. The Cretaceous cover of the Rheinische Masse contains no saline formations, but these set in in the Trias, immediately north of the Niedersächsische Uferrand. Where there is no saline intercalation, the Mesozoic mantle apparently adhered firmly to the north-moving substructure, abutted against and was thrust under the masses farther north, which had the resistence of the enormous mass of 8000—9000 metres of sediments of the North-German Basin behind them. In this underthrusting saline intercalations, notably in the Röt

and the Muschelkalk, seem to have acted as a lubricated shearzone (LOTZE).

In the deeper parts of the Basin of Münster the massif itself began to yield to the stress. Late-Saxonic deformation encroaches upon the Rheinische Masse whereever the Cretaceous cover reaches sufficient thickness to induce lability in the formerly rigid substructure. Already in the neighbourhood of the city of Münster there is indication of a gentle WNW-striking fold in the Cretaceous (**1**). In this northern area some increased activity is also observable within the Cretaceous along the NW cross-faults of the Variscan structure, which farther south do not affect the Cretaceous. Especially west of a line over Recklinghausen and Haltern, and north of Hamborn-Gelsenkirchen, late-Saxonic deformation sets in, as described bij BREDDIN (**2**). The movement shows itself as a series of WNW-striking folds, which gradually gain in importance in a north-westerly direction. Towards the southeast these folds turn into faultfolds and faultblocks. In the course of this process a number of the older Kimmeric NW-crossfaults were reactivated and show considerable throw within the Cretaceous. They also show the peculiarity that the movement was temporarily reversed. The older Kimmeric movements along these faults caused horsts and graben in the Permo-Triassic, and a still increased throw in the Coalmeasures indicates that activity dated already from the Paleozoic Asturic phase. These older movements were apparently a result of crustal stretching in SW-NE direction, possibly connected with the convexity of the Variscan arc. In the Cretaceous, on the contrary, compression to the N and NE is in evidence: normal faults turn into upthrusts; the graben in the older formations now turn into Cretaceous horsts, and anticlinal folds overlie older synclines or fault-troughs, and vice-versa. It must be noted that this reversal of motion is contemporaneous with the underthrust of the Osning and the other mentioned indications of suddenly increased northward creep of the substructure.

Still a little farther to the north, in the direction of the uplifts under Winterswyk, these movements continue to increase in importance (Raesfelder Kreidehorst). South of the Winterswyk block E-W folds set in, also involving upper-Cretaceous. Similar structures pierce the recent covering at Ochtrup, Bentheim and Losser. The entire anticlinorium of the Teutoburgerwald is explained by this mechanism. Beyond the North Sea it repeats itself in he Wolds of Yorkshire. It must be assumed to have been far more active and to have caused intense deformation in the deep, and presumably increasingly labile subsurface of the Central Netherlands and the North Sea. (**13**)

North of the Piesberg axis in the Teutoburger Wald a fairly general regional dip continues towards the north in the cuestas of the Wiehen and Weser Gebirge. The last visibly exposed fold is the Rehburg axis, running across the Dümmer See, eastward towards the Weser and the Steinhuder lake. Kimmeric folding, however, is no longer in evidence

on this axis, the only observable phase is Subhercynic (upper-Senonian unconformable over older Mesozoics); some further movement is indicated in the Miocene.

Still farther towards the north substructural movements are indicated, contemporaneous with the later-Saxonic phases, in the numerous salt-domes which dot the deep basin in Oldenburg. Periodic movements on the rising saline plugs prove that the same forces continue to be active in the deep subsurface. (**10**, **7**)

When we look towards the southwest, in Limburg and the Belgian Campine, along the eastern and northern boundary of the Brabant Massif, the Subhercynic movements become gradually obscured and more uncertain. As far as the Belgian Campine has been explored under the thick Tertiary and Cretaceous cover, Carboniferous Coalmeasures overlie the Silurian and older platform in a gentle north-dipping monocline, which has not been disturbed in late-Cretaceous time. These conditions have been ascertained as far to the northwest as the Woensdrecht boring, in western Noord-Brabant near Bergen-op-Zoom. Nowhere in this western region a sharp northern Abbruch is indicated, a condition similar to that prevailing in England. It is only at the eastern end of the Brabant Massif, in Limburg, that sharp downfaulting is in evidence. Here we are in the northern extension of the now greatly increased depression of the Kölnische Bucht. Now an enormous Tertiary graben develops, which can be traced all through the central Netherlands as far as the seacoast in Noord-Holland. The graben is caused by enormous NW-faults, which are clearly a continuation of those of the Kölnische Bucht. Now they cause, already between Sittard and Roermond, a depression of several thousands of metres. On the western side of this graben zone the bordering faults deflect to the WNW and finally almost West and seem to follow the northern contour of the Brabant Massif in the north-eastern Campine. A strong gravitational low is indicated a little farther north, with the same trend. The NW-faults originated already in Variscan time, were revived by the Kimmeric and the Subhercynic phases, and became very active again in Pliocene and Pleistocene time. They are still moving as proven by the epicenters of frequent earthquakes. These faults show the same reversal of motion during the Subhercynic phase as those described for Westphalen.

In the eastern Campine we have a curious instance of a northward dipping flat thrustfault in the Coalmeasures. It was encountered in the workings of the Limbourg-Meuse colliery at Eysden. Here a N-30°-W striking thrustfault (Faille d'Eysdenbosch) has become proven over a distance of some 1400 m along the strike. The fault dips only 26° to the NE; the southward movement along its plane amounts to 200 m (**5**). Less important rolls and crushing of a related nature have been described for the coalseams in the Campine mines. This movement, in this abnormal direction, can only be explained as underthrusting and the effect of a

northeastward creep in the Brabant substructure against the resistance of the deep sediment-filled basin of the Netherlands. The great border-fault Faille de Rothem, and probable further stepfaults succeeding it towards the north, bringing in Trias and Lias on the downthrown blocks, might constitute a kind of Abbruch. Whether any thrusting occurs in connection with these faults is unknown, but it would not be improbable that they dipped to the south. Not very much farther north Saxonic folding should begin to make an appearance in the deeply buried subsurface. It would seem however as if the main line of Saxonic disturbance should be traced much farther to the north and that the pre-Variscan Brabant mass, which already in Asturic time proved such an unyielding buttress, continued to intercept the pressures from the south.

Late-Pliocene and Pleistocene movements of an epeirogenetic character have caused the present topography of the Alpine foreland. These important changes of level are especially conspicuous in the region adjacent to the northern plains, but they occurred all over western Europe, including England. One of the most striking ones is the uplift of the barrier of the Hunsrück-Taunus across the course of the Rhine. Some uplift of the Taunus may have started in the Oligocene, cutting off marine communication from the north, but the upwarping of the Hunsrück and the Eifel culminated in the Pleistocene, forcing the Rhine to dig its 200 m deep canyon through the dam. Around the canyon between Bingen and Andernach upper-Pliocene gravels (Kieseloolithe) occur around 300 m above sealevel, and near Cologne at + 115 m. The recent lower terrace of the Rhine is at + 54 m at Andernach. Consequently there was a surelevation of the Hunsrück-Taunus of some 250 m since the Pliocene. An older-Tertiary peneplain on the Hunsrück, covered by presumably Oligocene gravels, occurs at + 600—700 m; in the Siegerland there is a peneplain of about the same elevation. The base of the upper-Cretaceous now lies at + 380 m at Aachen, but is uplifted to + 600 m on the immediately adjacent Hohe Venn block.

Farther west the Diestien (lowest Pliocene) lies at + 160—170 m at Calais and Ypres, but only at + 7 m at Antwerpen and at — 66 m at Woensdrecht, — 142 m. at Rosendaal, whilst a boring at Utrecht proves that it lies below — 365 m there.

In the Harz Mountains an old-Tertiary, possibly Eocene peneplain occurs now at + 1000 m. A lack of Tertiary sediments in the immediate vicinity does not permit further precision, but remnants of Tertiary in the Hessian depression suggest that a Tertiary sequence of about 1000 m has been eradicated there by erosion. It is very evident, therefor, that the foreland has become very severely warped in extremely recent, practically the present time.

The tectonic features which have been discussed are all the expression

of the periodically pulsating northward pressure on the foreland from the encroaching continental block that caused the Alpine orogeny. Entirely independent from these structures an autonomous line of deep N—S fractures traverses the entire western portion of the European continent from the mouths of the Rhone in the south to — possibly — the Mjösen lake in Scandinavia. Part of these N—S to NNE lines are well indicated on the map of figure 2, but the complete event can be better brought forward on the more general tectonic map of figure 3.

It is characteristic of all these fractures that they cause more or less wide and deep N—S fault-troughs and that they originate in the middle-Tertiary, notably in the Oligocene, but they continue to sink through all or most of the upper-Tertiary, the Pleistocene, and even recent time. The structure is entirely independent from other tectonic lines that cross it. The graben are only rarely deflected around massifs, but mostly split them indiscriminately. They seem to follow, however, some older — possibly very old — lines of weakness in the same N—S trend (3). They are traversed and continue to be affected by uplift or depression of the blocks they cross, which do not seem to lose their coherence. The importance and depth of the fracturing are emphasized by the fact that the fissures descend into depths where they open passage to the subcrustal basaltic magma. Active late-Tertiary volcanism characterizes the rifts: the greatest outpourings of basic lava's of this part of Europe occur along these lines. Like in Africa, a rifting of this nature does not cause a single clean-cut fracture, but — at least the surface expression — is a rather complicated zig-zag system of faults, fault troughs and even uplifts. The latter occur almost regularly in the zones bordering the fault-troughs at either side. Sometimes the troughs are duplicated by other more or less parallel rents; a zone of median horsts is almost the rule. The fractures are usually clearest where the crystalline, or at least substructural basement they have split is exposed at the surface in massifs.

The principal rift begins at the mouths of the Rhone in the Camargue and at Cape d'Agde on the Mediterranean coast. Amongst other depressions it causes the Rhone Valley trough, which is partly obscured by the foothills of the Alps of Dauphiné and farther north by the inroad of the Jura Mountains; it was generally deformed together with the Molasse foredeep of the western Alps, structures which were, to a considerable extent, caused or reworked by the Pliocene phases of the Alpine orogeny, a long time after the mid-Tertiary opening of the rift. The general area of the southern development is comprised between the old Moldanubian block of the Cevennes and the Plateau Central in the west, and in the east the Monts Maures of Var and the autochtonous crystalline basement blocks of the western Alps, from the Mercantour and Belledonne to the Mont Blanc. A parallel line of important fissures a little farther west rents the heart of the kata-metamorphic massif of the

TERTIARY RIFTING IN WESTERN EUROPE

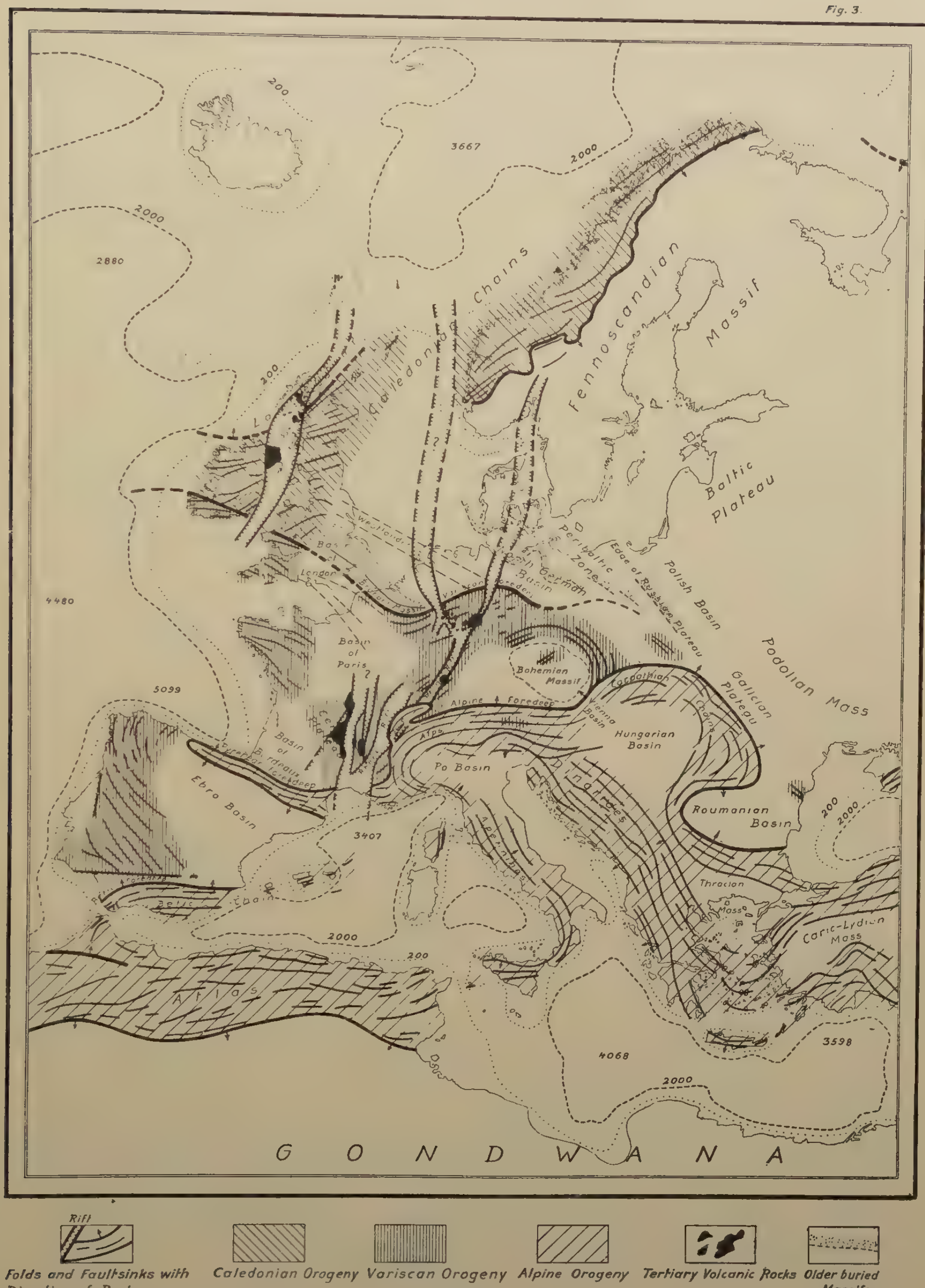


Fig. 3.

Plateau Central in the Loire fault-trough, and, more pronounced still, in the very deep depression of the Limagne, which is traceable in the sedimentary cover of the deep Basin of Paris as far north as Montargis, pointing in the direction of the deeper axis of the basin. A still more western fracture can be traced through the Plateau Central from along the Aveyron river and Villefranche, in an almost straight NNE direction, as far as Moulins in the Bourbonnais.

The Rhone Valley shows a maximum depression in the Oligocene, amounting to some 600 m. In the Miocene the bordering blocks, notably the Cevennes and the Beaujolais Mountains, were uplifted. The depth of the fault trough is not excessive, because in the transverse swell of Givors-Verpillière, borings easily reached the coalmeasures of the St. Étienne Basin, close to the Jura Mountains. In the Pliocene phases of the Alps the valley was uplifted instead of further depressed; in the region of Lyon the river had to cut a 60 metres deep channel.

The Rhone Valley lacks conspicuous volcanism. The rift of the Limagne, on the contrary, is a very deep chasm, where up to 1400 metres of Oligocene deposits are reported. The uplifting of the Auvergne set in during or shortly after the Miocene: an old-Miocene peneplain now lies at + 970 m. The great volcanism of the Puys reaches a maximum in the Pliocene, but the basalts of deep subcrustal origin are mainly Miocene. According to GLANGEAUD 1500 million cubic metres were ejected.

It seems a general characteristic that volcanism in these rifts becomes particularly active where they traverse old basement blocks; it becomes less conspicuous in labile depressions. It appears at the surface slightly later than the rifting.

In the north the Jura Mountains almost totally obscure the rift, but NNE-faults remain a conspicuous feature. Recent geophysical work in the Jura Mountains of Neuchâtel indicated the prevailence of tectonic lines in the same direction in the subsurface, crossing the NE trend of the Jura folds. This leads to where, between Belfort and Basel, the rift suddenly reappears in the great classic fault-trough of the Upper-Rhine Valley Graben.

The Upper-Rhine Valley possibly follows an old line of weakness suggested by Variscan shearing and Permo-Triassic depression. The region, however, was apparently unbroken in Jurassic and Cretaceous time, when it was simply part of the extensive high over southwestern Germany. The present rift dates from the Oligocene. In the middle-Oligocene the sea invaded the long and narrow trough both from the north and the south. The sinking was interrupted in the transition period between upper-Oligocene and Miocene, but became especially active again in upper-Pliocene—Pleistocene time. The bordering blocks were uplifted. The graben shows a certain asymmetry to the east: the most active faults were on the eastern rim; near Heidelberg they dip towards the mountain. Lateral shearing is indicated, suggesting a relative southward drift of the

western rim. At Durlach, east of Karlsruhe, the depression is reported to amount to 3200 m (Jurassic at — 2400 m according to MOOS); 210 m would have gone down in the upper-Pliocene, 70 m in the Pleistocene. A zone of median horsts is indicated at many points in the graben, especially in the south.

Volcanism also appeared in upper-Miocene time (Kaiserstuhl basalt), and this again in an area where the rift split the formerly united block of the Vosges and the Schwarzwald.

In the north the deep fault trough is suddenly cut off sharply by the very recent uplift of the Hunsrück-Taunus ridge, but the rifting evidently continues across it in the depression of Hessen. It is most marked in the Leinethal Graben. The NNE-faults can be traced at many points through the northern plains and point towards a similarly directed trough in the Fennoscandian Shield at Oslo (the Mjösen fault-trough). The Mesozoic, and still more a Tertiary age of the trough north of Oslo is, however, uncertain. All we really know is, that there exists a N—S graben, containing lower-Permian, and possibly younger-Permian effusives, overlying Rotliegend. A much younger superstructure may have become wiped off by erosion on the rapidly rising Fennoscandian mass.

Although the rifting is less conspicuously expressed by subsidence in Hessen than in the Upper-Rhine Valley, deep fracturing of the crust is most evident. Where the rifting crosses the central-German blocks a very complicated structure results from the interference between Alpine *compression* towards the north, causing NW—SE block-folds and ridges, and E—W *stretching*, resulting in N—S trending rifting. It is even possible that parallel stresses have acted as far to the east as Franken (Frankische Furche). Most active volcanism typifies the rift in Hessen and the Rhön. The Vogelsberg is by far the largest volcano of Central Europe, covering 2500 square kilometres with basaltic effusives; very numerous smaller eruptive vents dot the country as far north as Kassel and the Reinhardswald.

Although the Vogelsberg marks the extension of the Rhine rift through Hessen and Hannover in the general direction of Oslo, certain other remarkable features point to the northwest in the direction of Cologne and the Kölnische Bucht. N—S faults are little in evidence here, but a zone of very active volcanism, with largely basaltic effusives, indicates certain deep seated connections between the Basin of Mainz and the Kölnische Bucht. They begin as small necks in the region of Rüdesheim and Wiesbaden, become quite conspicuous in the Westerwald and the Eifel, and continue as far as Bonn (Siebengebirge) and Siegburg. These activities date all the way from Miocene to late-Pleistocene (Laacher See explosion). The basalts are Miocene.

Another old-Mesozoic line of weakness leads from Luxembourg in N—S direction towards the Kölnische Bucht. It is this connection that is marked by the volcanism of the Eifel. The leading direction of the

faulting is here SW—NE, characteristic for the entire eastern rim of the Basin of Paris and the Hunsrück-Taunus ridge. Oligocene rifting contouring the Ardennes mass and the Brabant Massif may, however, play a part in the deep subsurface. Conditions which we can observe at the surface are far from conclusive. Anyhow the great central fault-trough of the Netherlands (Groote Slenk), notably from Roermond on, has all the earmarks of the other rifts, although the general direction swings off to the NNW and NW, possibly connected with the contouring of the Brabant mass. Movement was certainly very active in Oligocene time, when the Rhone-Rhine rift had its maximum activity. An enormous thickness of marine upper-Oligocene was proven in borings near Vlodrop, east of Roermond, and also on the western side near Maeseyck. Farther to the north we only know, similarly as in the Kölnische Bucht, that periodic downfaulting continued all through the later Tertiary and the Pleistocene; here the Oligocene was no longer reached in borings. It would, therefor, seem possible that this trough was a more western fracture of the same general nature as the rifts in the Plateau Central. A median line of horsts is indicated along the full length of the Köln-Netherlands trough, beginning in the Ville, continuing in the Vierssen horst and the Peel horst. A farther extension of this median line of uplifts can be traced over Mill (near Boxmeer) in the direction of Amsterdam (het Gooi). Pleistocene movements on NW faults are conspicuous all over the Kölnische Bucht and South Limburg; they increase enormously in importance towards the northwest. In the already mentioned borings near Vlodrop the following relative displacements were noted as between the Peel-horst and the graben :

at the top of the upper-Oligocene ;	515 m;
within the Pliocene :	288 m;
at the base of the Pleistocene :	226 m.

The main depression runs in the direction of Alkmaar, with the median high still clearly in evidence. (**13**)

Between Scotland and Ireland there exist indications of another similar rift, *the Minch Channel,* which has decided points of similarity with the great more eastern rift of Continental Europe. It has the same typical features: upturning of adjoining blocks, fissuring independently across older transverse massifs, and Tertiary volcanism. Most of the rift is presumably submerged under the Irish Sea and the Atlantic. Shears suggest a relative displacement of Ireland towards the south. Volcanism is very much in evidence. The Paleozoic and older platform is depressed below sealevel within the fault-trough; present high ground, including the Plateau of Antrim, consists mainly of Tertiary basalt, the flows of which reach aggregate thicknesses of between 500 and 1000 metres. Relicts of Mesozoic beds (upper-Cretaceous) exist in Antrim. The flows of lava

cannot be dated accurately, but it is suggested by plant remains that part of the tuffs may be of Eocene age, consequently this more western rift may have had an earlier origin than the Continental Oligocene ones. Similar volcanics as in Antrim, occur in Arran, on Mull and Skye, in Scotland, and other smaller islands within the Minch. On Mull, Skye and Eigg the basalts are also associated with Eocene leaf-beds, indicating that this area was let down between the Outer Hebrides and the mainland of Scotland. It is possible that the later phases of igneous activity were prolonged into subsequent divisions of Tertiary time.

Farther south in the Irish Sea no similar volcanism is known, the only point being the Wolf Rock, 30 km SW of the Land's End (phonolite). One has to go far out into the Atlantic west of Ireland to find similar effusives: Rockall, 380 km west of the Outer Hebrides, and the shoals of the Porcupine Bank, 275 km west of Ireland.

A great deal of controversy has existed whether these great rents in the continental mass of Europe are *rifts or ramps*, in other words whether these fault-throughs, with their upturned sideblocks, are due to subsidence — whether it be gravitational along the apex of an old arch, or through tension —, or whether (ramps) they are the result of compression, the downthrust center between two upthrust jaws, connoting, at least in part, a high angle attitude. The writer favors the true rift explanation. He argues that a structural feature of such enormous lateral dimensions (1800 km in length from the Rhone mouths to either Mjösen or the North Sea) cannot be explained in any other way than by westward stretching and fissuring of the continental block of Europe, similarly as we whitness in Africa. There also it is not a clean-cut single fracture, but a zone of fissures and fault-troughs of varying direction in details. The primary cause would have to be continental west-drift and the tearing open of the Atlantic. The relative smallness of the westward stretching as compared to the enormous compression caused by the northward drift of Gondwana, active through the entire upper-Paleozoic, and again through all later-Mesozoic—Tertiary times, should not close our eyes as to the primary importance of rifts of such length, associated with so much deep seated volcanism. Much of the argument has centred around more closely studied local features, like the Upper-Rhine Graben and Hessen. The problem cannot be put so simply, interesting though these studies are. Naturally these immense crustal fractures are very complicated, and what we observe is only the very superficial effect of a deep rent in the lower crust. They all originated in the Tertiary, but — although they are conspicuously autonomous — in certain areas they apparently follow lines of weakness. This would have to be expected, and is also the case in Africa. If westdrift, and resultant westward stretching are the primary cause, many other stresses and resistences in the complicated mosaic of the outer crust are certain to confuse the picture. Simultaneously there has occurred relative

N and NE drift of notable regions in the platform, setting up shearing and, quite probably, local reversal of stretching into temporary compressive forces. Then we have quite considerable, probably largely isostatic changes of level of several great crustal units to confuse us. In certain areas these movements affect parts of the rift-trough itself, and are apt to have effaced its surface character over a considerable expanse. Then the local reaction of the superficial crust to the rifting, which took place in the deep crystalline crust at some scores of kilometres below the surface, is naturally variable. In some places the fault-trough effect is very much more pronounced than elsewhere, in other places it is mainly volcanic activity (deep basic subcrustal magmas!) that indicates the deep rent in the crust, whilst the surface remains more or less undisturbed, or merely shows some sagging. The filling of the resultant void can take place either by sediments from above, or by magma from below. The masses of lava that poured out at the surface are sometimes enormous. It is impossible, therefor, to decide so simply, from a few local observations, whether the great Rhone-Mjösen fissure, with its many side branches, taken as a whole, is a rift or a ramp. A fracture of such dimensions is evidently caused by major crustal movements of a far greater order of magnitude than any of its local and superficial effects. These latter are only incidents, variable as to local conditions, and to be viewed and interpreted as such.

CONCLUSIVE SUMMARY.

The Variscan folding and intense overthrusting had caused great zonal duplication of relatively lighter crustal rocks and these areas have remained active isostatically. After general consolidation had progressed, following the Permo-Carboniferous revolution, the folded area had become a firm corrugated crust of great resistence, particularly to forces operating at an angle to the original strike. The rigidity, however, became locally weakened. This was particularly the case where the substructure became depressed under deep Kimmeric and older basins with more recent filling. The location of such basins may have been caused either by a preexisting local weakness in the crust, by isostasy, or by deepseated pressures from the initial stages of the Alpine orogeny. In the Mesozoic the northward creep was revived, but a greatly changed crust became subject to it on the Alpine foreland. Yielding became very differentiated and locally variable, in accordance with local substructural differences in rigidity. It is natural to expect the non-homogeneous crust to give way along shear lines, directed in the general direction of the creep, but now on a much larger scale than before it became solidified to its present extent. True folding would not anymore take place easily, but crustal shortening must principally have happened along strike faults, blockfolding, or broad basement-warping (undation). Folding would be confined to areas of enhanced lability in the deeper basins. Roughly N—S lines of weakness were caused,

which at first expressed themselves only as sagging. When finally the west of Europe became subject to Tertiary stretching in conjunction with the opening of the Atlantic, a few great rifts formed: first the Minch-Irish Sea Rift in the west, and shortly after the Rhone-Rhine-Mjösen Rift a little farther east, with a presumable great side branch traversing the Netherlands and the northern North Sea. The movements of the great crustal units between these major fractures became differentiated. There appears a natural tendency of the more western blocks to lag behind in the general creep towards the north, resulting in an apparent southward sliding of the western units relative to their eastern neighbors, and shear structure. Some blocks may have become rotated. The importance of the few major rifts as crustal fractures of a higher order is indicated by prolific outpouring of deep subcrustal magma at many points along their course (alternating of course with more acid products of later differentiation or melts caused by fusion). These effusives are most conspicuous on the geologic map where older crustal blocks were split open and the basement platform outcrops at the surface.

Broad zones of shearing were set up at many localities. Local variations in rigidity of older, probably deeply rooted masses like the Brabant Massif, or more solidified substructural units like the Rheinische Masse, the Ardennes and the Harz Mountains, cause interruption of the regularity of the picture, anomalies in the general trends and in the yielding to deformation. Northward drift is clearest expressed in the Rheinische Masse and the Harz, complicated by salt tectonics. Another set of structures runs through it all, a widespread fracturing of Continental Europe along a prevalent WNW—ESE direction, parallel to the south-western edge of the Russian Table. Possibly these also may be an effect of compression on a broad scale, warping and fracturing of the basement, under the influence of northward pressures from the two great orogenies, but deflected by a lateral resistence from the massive Russian Plateau. All this caused an extremely complicated mosaic of blocks, which by their relative displacements set up a multitude of local stresses and resultant minor deformation, obscuring *the primary pattern*.

This latter remains, *first* and principally: *regional crustal creep towards the north* into the triangular space between Eria and Fennoscandia, between the Russian Table in the east and the thoroughly consolidated Caledonide blocks in the west, with some minor rigid nuclei between: the Brabant Massif, the Ardennes and Rheinische Masse block, Moldanubia, the Plateau Central and Armorica. (Others may be hidden under the later filling of the deep basins). In this reentrant the successive Variscan and Alpine arcs had pushed themselves with increasing convexity towards the north. *Secondly:* autonomous N—S *rifting* of the entire West-European crust by continental west-drift, in conjunction with the opening of the Atlantic.

Wylre, February 1938.

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Physics. — *Magnetic storm and variation of cosmic rays.* By J. CLAY and E. M. BRUINS.

(Communicated at the meeting of February 26, 1938.)

During the magnetic storm of January 24—26 we observed the variation of cosmic rays in our recording instruments with ionization chambers of 40 l. with 45 Atm. argon. The chambers I and II were under a shield of 110 cm *Fe* and instrument III under 12 cm *Fe*.

In fig. 1 we see the situation of the three ionization chambers. The iron wall in the foreground in front of the chambers is taken away, to be able to see the chambers on the photograph.

We found a very considerable variation of the intensity of the rays during the storm. It was a disadvantage that in the same period the barometer changed so much, as it is always difficult to give the exact correction for this variation, because it is not the same in different cases. It may be on account of this that our correction is not quite the correct one, but the differences will not be more than 1 or 2 %.

The most remarkable result in this case is now in contrast [1]) with the storm of 26/4 '37, that we find in all three chambers an increase before the usual decrease, which is found in most cases correlated with magnetic storms, and secondly that also the instruments under 110 cm *Fe* are influenced.

In the instrument under 12 cm *Fe* there was at one moment an increase of about 3 % and then after that suddenly a decrease of about 9 %, a difference from the undisturbed value of 6 %.

The influence on the ionization in the chambers I and II is different, as we see in fig. 2. The scale on the left gives the value of the potential applied to the central electrode in order to compensate the ionization charge in one hour, but as the values of these capacities are not the same, the potentials also are different. In reality the ionization charge is nearly the same. Now the question is not yet solved how it is possible that the sensitivity of II to the magnetic variations is larger than that of I. We find a similar difference in the sensitivity to barometric variations.

We interchanged the instruments I and II and found that the sensitivity is correlated with the place and not with the instrument. The variations of I and II can be read from graph 2.

The variation in instrument I was small, only a very small maximum of 0.3 % above and a minimum of 2.4 % below the mean value.

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[1]) J. CLAY and E. M. BRUINS, Physica, 5, 111 (1938).

The variation of instrument II, a maximum of 2.3 % and a minimum of 3.3 %. The maximum of all three instruments was at 25 Jan.

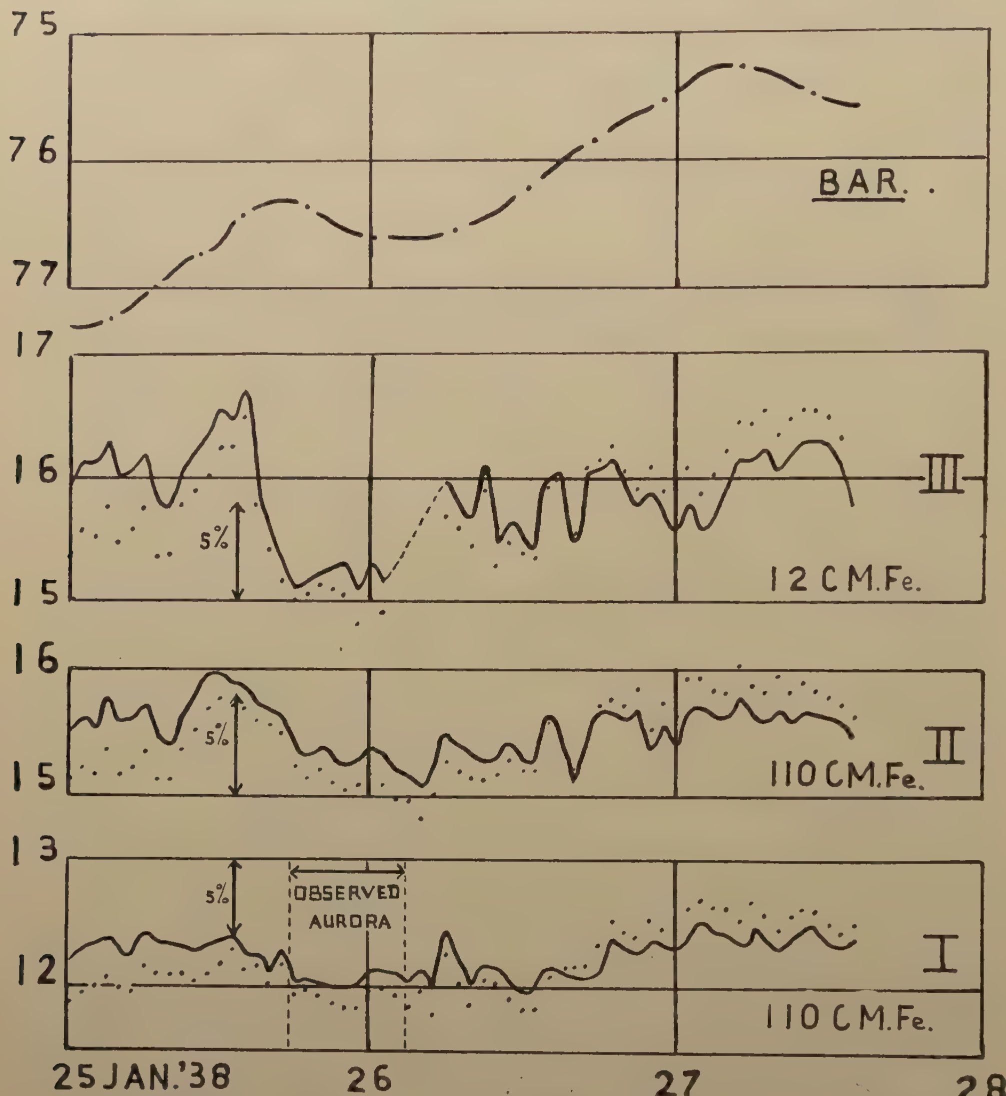

Fig. 2. The variations of cosmic-ray intensities during the magnetic storm of Jan. 24—26. The points in the graph are the recorded values; the full line gives the corrected values.

12 h. M. G. T. and the minima between 25 Jan. at 18 h. M. G. T. and 26 Jan. 4 h. M. G. T.

At these moments the barometer was the same, so that we have a good chance that these differences were not affected by influences of barometric changes. When we shall have collected the results of the magnetic observations during the same period, we will discuss the correlation and the causes.

J. CLAY AND E. M. BRUINS: MAGNETIC STORM AND VARIATION OF COSMIC RAYS.

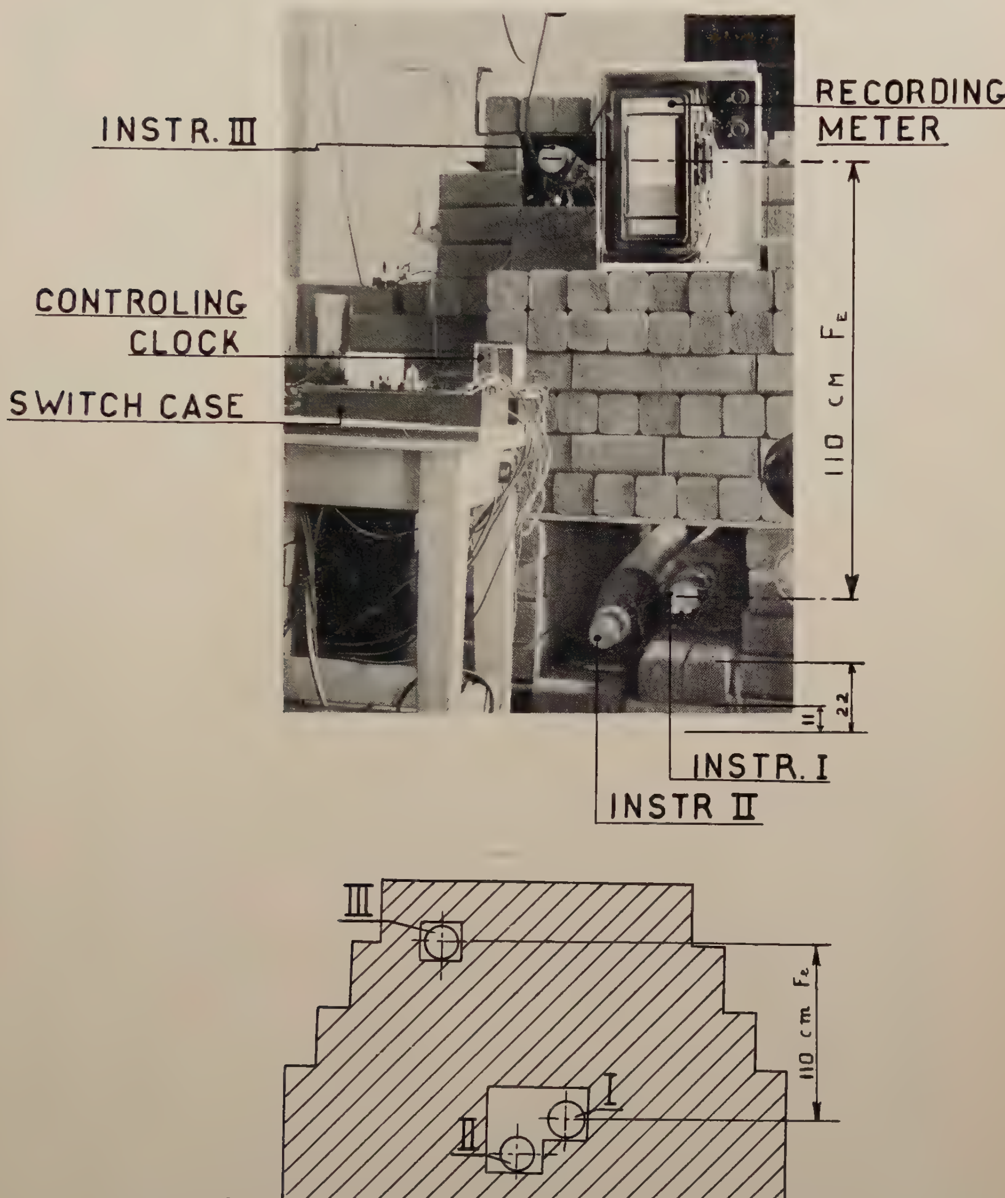

Fig. 1. The recording ionization instruments, under their iron shields.

Mathematics. — *Sur deux, trois ou quatre nombres premiers.* Par J. G. VAN DER CORPUT. (Quatrième communication).

(Communicated at the meeting of February 26, 1938.)

Lemme 20: *Posons*

$$H(q,t) = \frac{1}{\varphi^4(q)} \sum_{a;h_1,h_2,h_1',h_2'} e^{2\pi i \frac{a}{q}(\pm h_1^2 \pm h_2^2 \pm h_1'^2 \pm h_2'^2 - t)}, \quad . \quad . \quad . \quad (47)$$

où a, h_1, h_2, h_1', h_2' *parcourent les nombres naturels* $\leqq q$, *qui sont premiers avec* q *(donc* $H(q,t)$ *est la fonction, définie par (37)); soit* τ *le nombre des termes, figurant dans l'exposant et fourni du signe moins; donc* $\tau = 0, 1, 2, 3$ *ou* 4.

Pour tout nombre premier $w > 2$, *qui est un facteur de* t, *on a*

$$H(w,t) = \frac{w^2+6w+1}{(w-1)^3}, \text{ lorsque } w \equiv 1 \ (mod.\ 4);$$

$$= \frac{w^2-6w+1}{(w-1)^3}, \text{ lorsque } w \equiv -1 \ (mod.\ 4), \quad \tau = 0 \text{ ou } 4;$$

$$= -\frac{w+1}{(w-1)^2}, \text{ lorsque } w \equiv -1 \ (mod.\ 4), \quad \tau = 1 \text{ ou } 3;$$

$$= \frac{(w+1)^2}{(w-1)^3}, \text{ lorsque } w \equiv -1 \ (mod.\ 4), \quad \tau = 2.$$

Pour tout nombre premier $w \equiv 1$ *(mod. 4), qui n'est pas un facteur de* t, *on a*

$$H(w,t) = -\frac{5w^2+10w+1}{(w-1)^4} \quad \text{ou} \quad \frac{3w+1}{(w-1)^3},$$

selon que t *est un reste quadratique de* w *ou non.*

Pour tout nombre premier $w \equiv -1$ *(mod. 4), qui n'est pas un facteur de* t, *on a*

$$H(w,t) = -\frac{5w^2-10w+1}{(w-1)^4}, \text{ lorsque } \tau = 4 \text{ et } t \text{ est un reste quadratique de } w;$$
$$\text{aussi lorsque } \tau = 0 \text{ et } t \text{ est un non-reste de } w;$$

$$= \frac{(w+1)(3w-1)}{(w-1)^4}, \text{ lorsque } \tau = 0 \text{ ou } 3 \text{ et } t \text{ est un reste quadratique de } w;$$
$$\text{aussi lorsque } \tau = 1 \text{ ou } 4 \text{ et } t \text{ est un non-reste de } w;$$

$$H(w,t) = -\frac{(w+1)^2}{(w-1)^4}$$, *lorsque* $\tau = 2$;
aussi lorsque $\tau = 1$ *et* t *est un reste quadratique de* w:
aussi lorsque $\tau = 3$ *et* t *est un non-reste de* w.

Remarque: On a $1 + H(3, t) = 0$ dans les cas suivants:

1). $\tau = 0$; $t \equiv 0$ ou -1 $(mod.\ 3)$;
2). $\tau = 1$; $t \equiv 0$ ou 1 $(mod.\ 3)$;
3). $\tau = 2$; $t \equiv 1$ ou -1 $(mod.\ 3)$;
4). $\tau = 3$; $t \equiv 0$ ou -1 $(mod.\ 3)$;
5). $\tau = 4$; $t \equiv 0$ ou 1 $(mod.\ 3)$;

c'est-à-dire $1 + H(3, t) = 0$, si $t \not\equiv 1 - 2\tau$ $(mod.\ 3)$. Si par contre $t \equiv 1 - 2\tau$ $(mod.\ 3)$, on a $1 + H(3, t) > 0$.

Chaque nombre premier $w > 3$ satisfait pour tout entier t à l'inégalité $1 + H(w, t) > 0$.

Démonstration: Pour tout nombre premier $w > 2$ on a

$$H(w,t) = (w-1)^{-4} \sum_{a=1}^{w-1} e^{-\frac{2\pi i a t}{w}} \sum_{h_1=1}^{w-1} e^{\frac{2\pi i a \theta_1 h_1^2}{w}} \sum_{h_2=1}^{w-1} e^{\frac{2\pi i a \theta_2 h_2^2}{w}}$$
$$\sum_{h_1'=1}^{w-1} e^{\frac{2\pi i a \theta_3 h_1'^2}{w}} \sum_{h_2'=1}^{w-1} e^{\frac{2\pi i a \theta_4 h_2'^2}{w}},$$

où $\theta_\lambda^2 = 1$, donc d'après le lemme 16

$(w-1)^4 H(w,t)$

$$= (1-\sigma_1\sqrt{w})(1-\sigma_2\sqrt{w})(1-\sigma_3\sqrt{w})(1-\sigma_4\sqrt{w}) \sum_{\substack{a=1 \\ a \text{ reste de } w}}^{w-1} e^{-\frac{2\pi i a t}{w}}$$

$$+ (1+\sigma_1\sqrt{w})(1+\sigma_2\sqrt{w})(1+\sigma_3\sqrt{w})(1+\sigma_4\sqrt{w}) \sum_{\substack{a=1 \\ a \text{ non-reste de } w}}^{w-1} e^{-\frac{2\pi i a t}{w}};$$

j'ai posé $\sigma_\lambda = 1$ ou $i\,\theta_\lambda$, selon que w est congru à $+1$ ou -1 modulo 4.

Distinguons deux cas différents.

1. Soit le nombre premier $w > 2$ un facteur de t. Alors on a

$$2(w-1)^3 H(w,t) = (1-\sigma_1\sqrt{w})(1-\sigma_2\sqrt{w})(1-\sigma_3\sqrt{w})(1-\sigma_4\sqrt{w})$$
$$+ (1+\sigma_1\sqrt{w})(1+\sigma_2\sqrt{w})(1+\sigma_3\sqrt{w})(1+\sigma_4\sqrt{w}).$$

Lorsque $w \equiv 1$ $(mod.\ 4)$, les quatre nombres σ_λ sont égaux à 1, donc

$$(w-1)^3 H(w,t) = \tfrac{1}{2}(1-\sqrt{w})^4 + \tfrac{1}{2}(1+\sqrt{w})^4 = w^2 + 6w + 1.$$

Dans le cas où $w \equiv -1$ (*mod.* 4), $\tau = 0$ ou 4, les quatre nombres σ_λ sont égaux (notamment égaux à $\pm i$), par conséquent

$$(w-1)^3 H(w, t) = \tfrac{1}{2}(1-i\sqrt{w})^4 + \tfrac{1}{2}(1+i\sqrt{w})^4 = w^2 - 6w + 1.$$

Dans le cas où $w \equiv -1$ (*mod.* 4), $\tau = 1$ ou 3, trois des nombres σ_λ sont égaux, et on a

$$\begin{aligned}(w-1)^3 H(w, t) &= \tfrac{1}{2}(1-i\sqrt{w})^3(1+i\sqrt{w}) + \tfrac{1}{2}(1+i\sqrt{w})^3(1-i\sqrt{w}) \\ &= -(w-1)(w+1).\end{aligned}$$

Dans le cas où $w \equiv -1$ (*mod.* 4), $\tau = 2$, deux des nombres σ_λ sont égaux à i, tandis que les deux autres sont égaux à $-i$, donc

$$(w-1)^3 H(w, t) = (1-i\sqrt{w})^2(1+i\sqrt{w})^2 = (w+1)^2.$$

2. Supposons que l'entier t ne soit pas divisible par le nombre premier impair w. Le lemme 16 nous apprend

$$\begin{aligned}&2(w-1)^4 H(w, t) = \\ &\quad -(1-\sigma_1\sqrt{w})(1-\sigma_2\sqrt{w})(1-\sigma_3\sqrt{w})(1-\sigma_4\sqrt{w})(1-\sigma\sqrt{w}) \\ &\quad -(1+\sigma_1\sqrt{w})(1+\sigma_2\sqrt{w})(1+\sigma_3\sqrt{w})(1+\sigma_4\sqrt{w})(1+\sigma\sqrt{w}).\end{aligned}$$

Dans le cas où $w \equiv 1$ (*mod.* 4) et t est un reste quadratique de w, les nombres $\sigma_1, \ldots, \sigma_4, \sigma$ sont égaux à 1, donc

$$(w-1)^4 H(w, t) = -\tfrac{1}{2}(1-\sqrt{w})^5 - \tfrac{1}{2}(1+\sqrt{w})^5 = -5w^2 - 10w - 1.$$

Dans le cas où $w \equiv 1$ (*mod.* 4) et t est un non-reste de w, les quatre nombres σ_λ sont égaux à 1, tandis que $\sigma = -1$, donc

$$\begin{aligned}&(w-1)^4 H(w, t) \\ &= -\tfrac{1}{2}(1-\sqrt{w})^4(1+\sqrt{w}) - \tfrac{1}{2}(1+\sqrt{w})^4(1-\sqrt{w}) = (w-1)(3w+1).\end{aligned}$$

Considérons maintenant les nombres premiers $w \equiv -1$ (*mod.* 4). Dans le cas où $\tau = 0$ et t est un non-reste de w, et dans le cas où $\tau = 4$ et t est un reste quadratique de w, les cinq nombres $\sigma_1, \ldots, \sigma_4, \sigma$ sont égaux (notamment égaux à $\pm i$), par conséquent

$$(w-1)^4 H(w, t) = -\tfrac{1}{2}(1-i\sqrt{w})^5 - \tfrac{1}{2}(1+i\sqrt{w})^5 = -5w^2 + 10w - 1.$$

Dans le cas où $\tau = 0$ ou 3 et t est un reste quadratique de w, et dans le cas $\tau = 1$ ou 4 et t est un non-reste de w, le système $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma$ contient quatre nombres égaux (notamment égaux à $\pm i$), tandis que le cinquième nombre possède la valeur opposée, donc

$$\begin{aligned}(w-1)^4 H(w, t) &= -\tfrac{1}{2}(1-i\sqrt{w})^4(1+i\sqrt{w}) - \tfrac{1}{2}(1+i\sqrt{w})^4(1-i\sqrt{w}) \\ &= (w+1)(3w-1).\end{aligned}$$

Dans le cas où $\tau = 2$; dans le cas où τ est 1 et t un reste quadratique de w, et dans le cas où τ est 3 et t un non-reste de w, le système $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma$ contient trois nombres égaux à $\pm i$, tandis que les deux autres nombres sont égaux à $\mp i$, par conséquent

$$\begin{aligned}(w-1)^4 H(w, t) &= -\tfrac{1}{2}(1-i\sqrt{w})^3(1+i\sqrt{w})^2 - \tfrac{1}{2}(1+i\sqrt{w})^3(1-i\sqrt{w})^2 \\ &= -(w+1)^2.\end{aligned}$$

Lemme 21: *Sous les conditions du lemme précédent on a*

$$H(2, t) = (-1)^t;$$

$$H(4, t) = 0 \ \textit{ou} \ (-1)^{\frac{2\tau+t}{2}}\, 2,$$

selon que t est impair ou pair;

$$H(8, t) = (-1)^{1+\frac{2\tau+t}{4}}\, 4 \ \textit{ou} \ 0,$$

selon que $2\tau + t$ est divisible par 4 ou non.

Démonstration: La formule (47) nous apprend

$$H(2, t) = e^{-\pi i t} = (-1)^t.$$

En vertu de

$$\sum_{h=1,3} e^{\pm\frac{2\pi i a h^2}{4}} = 2 e^{\pm\frac{2\pi i a}{4}} \qquad (a \text{ impair})$$

on obtient

$$H(4, t) = \frac{2^4}{2^4} \sum_{a=1,3} e^{\frac{2\pi i a}{4}(4-2\tau-t)} = 0 \ \text{ou} \ (-1)^{\frac{2\tau+t}{2}}\, 2,$$

selon que t est impair ou pair. La relation

$$\sum_{h=1,3,5,7} e^{\pm\frac{2\pi i a h^2}{8}} = 4 e^{\pm\frac{2\pi i a}{8}} \qquad (a \text{ impair})$$

nous apprend

$$H(8, t) = \frac{4^4}{4^4} \sum_{a=1,3,5,7} e^{\frac{2\pi i a}{8}(4-2\tau-t)} = (-1)^{1+\frac{2\tau+t}{4}}\, 4 \ \text{ou} \ 0,$$

selon que $2\tau + t$ est divisible par 4 ou non.

Remarque: Si $2\tau + t \equiv 4$ (*mod.* 8), on a par conséquent

$$H(2, t) = 1, \qquad H(4, t) = 2, \qquad H(8, t) = 4,$$

donc

$$1 + H(2, t) + H(4, t) + H(8, t) = 8.$$

Dans l'autre cas on a

$$1 + H(2, t) + H(4, t) + H(8, t) = 0;$$

en effet, le membre de gauche est égal à

$$\begin{aligned} 1 - 1 + 0 + 0 &= 0 \quad \text{lorsque} \quad 2\tau + t \equiv 1 \quad (mod.\ 2), \\ 1 + 1 - 2 + 0 &= 0 \quad \text{lorsque} \quad 2\tau + t \equiv 2 \quad (mod.\ 4), \\ 1 + 1 + 2 - 4 &= 0 \quad \text{lorsque} \quad 2\tau + t \equiv 0 \quad (mod.\ 8). \end{aligned}$$

Lemme 22: *Si le polynôme $\psi(x)$ du $g^{ième}$ degré prend une valeur entière pour tout entier x, les coefficients du polynôme $g!\,\psi(x)$ sont tous entiers.*

Démonstration: Posons

$$\psi(x) = b_0 + b_1 \binom{x}{1} + \ldots + b_g \binom{x}{g},$$

où $\binom{x}{1}, \binom{x}{2}, \ldots, \binom{x}{g}$ représentent des coefficients binomiaux. Les nombres $b_0, b_1, \ldots, b_g$ peuvent être définis de la façon suivante

$$b_0 = \psi(0),$$

$$b_0 + b_1 = \psi(1),$$

$$b_0 + \binom{2}{1} b_1 + b_2 = \psi(2),$$

$$\cdots\cdots\cdots\cdots$$

$$b_0 + \binom{g}{1} b_1 + \ldots + \binom{g}{g-1} b_{g-1} + b_g = \psi(g).$$

Il résulte de ces relations que $b_0, b_1, \ldots, b_g$ sont entiers, donc que les coefficients du polynôme $g!\,\psi(x)$ sont également tous entiers.

Lemme 23: *Si le polynôme $\psi(x)$ du degré précis $g \geqq 1$ prend une valeur entière pour toute valeur entière de x, si le nombre naturel q n'est divisible ni par un carré impair > 1, ni par 16, on a pour tout nombre $Z > 3$ et pour tout entier s*

$$\sum_{|x| \leqq Z}{}' \ (q, \psi(x) + s) < c_{35}\, \tau^{\lambda+1}(q) \{ Z + Min\,(q, Z^g + |s|) \}; \quad (48)$$

Σ' est étendu à tous les entiers x tels que $|x| \leqq Z$ et $\psi(x) + s \neq 0$; (q, t) est le plus grand commun diviseur de q et t; c_{35} dépend uniquement du choix du polynôme $\psi(x)$; $\tau(q)$ désigne le nombre des diviseurs de q;

$$\lambda = \frac{\log g}{\log 2};$$

Min (q, t) désigne le plus petit des deux nombres q et t.

Démonstration: Nous nous demandons d'abord combien de fois tout au plus un nombre naturel donné d figure dans le système formé par les nombres $(q, \psi(x)+s)$, où x parcourt tous les entiers x tels que $|x| \leqq Z$ et $\psi(x)+s \neq 0$. Seuls les diviseurs d de q entrent en considération et un diviseur de q n'est divisible ni par un carré impair >1 ni par 16.

D'après le lemme précédent tous les coefficients du polynôme $g!\,8(\psi(x)-\psi(0))$ sont divisibles par 8. Le plus grand commun diviseur D de ces coefficients, qui dépend uniquement du choix du polynôme $\psi(x)$, est donc divisible par 8. Posons $d=\delta\, w_1 w_2 \ldots w_s$, où $\delta=(d, D)$, (donc $\delta \leqq D$) et $w_1, w_2, \ldots, w_s$ désignent des nombres premiers.

D possède tout au moins trois facteurs 2, tandis que d possède tout au plus trois facteurs 2; par conséquent δ et d possèdent autant de facteurs 2, de sorte que $w_1, w_2, \ldots, w_s$ sont impairs. Parce que d n'est pas divisible par un carré impair >1, les nombres $w_1, w_2, \ldots, w_s$ sont différents l'un de l'autre, et δ, donc également D, n'est divisible par aucun des nombres $w_1, w_2, \ldots, w_s$.

La relation $d=(q, \psi(x)+s)$ implique

$$g!\,\psi(x)+g!\,s \equiv 0 \qquad (mod.\ w_\sigma). \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Les coefficients du polynôme $g!(\psi(x)-\psi(0))$ sont entiers: au moins un de ces coefficients n'est pas divisible par w_σ, parce que D n'est pas divisible par w_σ. Un système formé de w_σ entiers consécutifs contient par conséquent tout au plus $g=2^\lambda$ entiers x qui satisfont à (49). Un système formé de $w_1 w_2 \ldots w_s$ entiers consécutifs contient donc tout au plus $2^{\lambda s}$ entiers x qui satisfont simultanément au système des s congruences

$$g!\,\psi(x)+g!\,s \equiv 0 \qquad (mod.\ w_\sigma) \ \ (\sigma=1, 2, \ldots, s).$$

On a

$$2^s=\tau(w_1 w_2 \ldots w_s) \leqq \tau(q).$$

Un système formé de d entiers consécutifs contient donc tout au plus $D\,\tau^\lambda(q)$ entiers x qui satisfont simultanément aux s congruences nommées, donc tout au plus $D\,\tau^\lambda(q)$ nombres x tels que $(q, \psi(x)+s)=d$. Le nombre des entiers x tels que $|x| \leqq Z$ et $(q, \psi(x)+s)=d$ est par conséquent tout au plus égal à

$$D\tau^\lambda(q)\left(\frac{1+2Z}{d}+1\right) < 3D\,\tau^\lambda(q)\left(\frac{Z}{d}+1\right).$$

La contribution de ces entiers x au membre de gauche de (48) est donc inférieure à

$$3D\,\tau^\lambda(q)(Z+d) < c_{35}\,\tau^\lambda(q)\ \ (Z+Min(q, Z^g+|s|));$$

en effet, les relations $d=(q, \psi(x)+s)$ et $\psi(x)+s \neq 0$ entraînent

$$3Dd \leqq 3Dq \leqq c_{35}\,q \ \text{ et } \ 3Dd \leqq 3D\,|\psi(x)+s| \leqq c_{35}(Z^g+|s|).$$

Par conséquent, le membre de gauche de (48) est tout au plus égal à

$$c_{35}\,\tau^{\lambda}(q)\,(Z + Min\,(q, Z^{g} + |s|))\sum_{d/q} 1$$

et cette dernière expression est égale au membre de droite de (48).

Lemme 24: *Si le polynôme $\psi(x)$ du degré précis $g \geqq 1$ prend une valeur entière pour toute valeur entière de x, si ε est positif, $Z > 3$, s entier, et Q désigne un nombre naturel, chacune des fonctions $H(q, t)$, définies par* (35), (36) *et* (37), *satisfait à l'inégalité*

$$\sum_{x \leqq Z}{}' \sum_{q\ Q+1}^{\infty} |H(q, \psi(x) + s)| < c_{36}\{ZQ^{\varepsilon-1} + Z^{\varepsilon} + |s|^{\varepsilon}\}; \tag{50}$$

Σ' est étendu à tous les entiers x tels que $|x| \leqq Z$ et $\psi(x) + s \neq 0$; c_{36} dépend uniquement de ε et du choix du polynôme $\psi(x)$.

Démonstration: Les lemmes 18, 19 et 20 nous apprennent pour tout nombre premier $w > 2$ et pour tout entier t

$$|H(w, t)| < \frac{2^5}{w} \text{ ou } < \frac{2^5}{w^2},$$

selon que w est un facteur de t ou non. Pour tout nombre naturel q qui est divisible par un carré impair > 1 ou par 16, on a $H(q, t) = 0$ d'après le lemme 17. Pour les autres nombres naturels q le lemme 15 nous apprend

$$H(q, t) = H(2^{\omega}, t) \prod_{\substack{w/q \\ w>2}} H(w, t);$$

le produit est étendu à tous les facteurs premiers impairs w de q, tandis que ω désigne le nombre des facteurs 2 de q (donc $\omega = 0, 1, 2$ ou 3). Par conséquent

$$|H(q, t)| \leqq c_{37} \prod_{\substack{w/q \\ w>2}} \frac{2^5}{w^2} \cdot \prod_{\substack{w/q \\ w/t \\ w>2}} w$$

$$\leqq c_{38} \frac{\tau^5(q)}{q^2}(q, t),$$

où c_{37} et c_{38} sont des constantes absolues. Le membre de gauche de (50) est donc

$$\leqq c_{38} \sum_{q=Q+1}^{\infty} \frac{\tau^5(q)}{q^2} \sum_{|x| \leqq Z}{}' (q, \psi(x) + s)$$

$$\leqq c_{38}\, c_{35} \sum_{q=Q+1}^{\infty} \frac{\tau^{6+\lambda}(q)}{q^2}\{Z + Min\,(q, Z^{g} + |s|)\}$$

d'après le lemme précédent, donc

$$< c_{39} \left\{ \sum_{q=Q+1}^{\infty} q^{\varepsilon-2} Z + \sum_{Q<q\leqq Z^g+|s|} q^{\frac{\varepsilon}{g}-1} + \sum_{q>Z^g+|s|} q^{\frac{\varepsilon}{g}-2} (Z^g+|s|) \right\}$$

$$< c_{40} \{ Z Q^{\varepsilon-1} + Z^{\varepsilon} + |s|^{\varepsilon} \};$$

c_{39} et c_{40} dépendent uniquement de ε et du choix du polynôme $\psi(x)$.

Proposition 3: *Sous les conditions de la proposition* 2 (troisième communication, p. 97), *chaque nombre* $Z > 3$ *tel que*

$$Max\,(B, B', B_1^2, B_2^2, B_1'^2, B_2'^2) \leqq \tfrac{1}{4} Z^g, \quad . \quad . \quad . \quad . \quad . \quad (51)$$

chaque entier s *et chaque nombre naturel* m *satisfont à l'inégalité*

$$\sum_{|x|\leqq Z}{}' \; | F(\psi(x)+s) - \Phi(\psi(x)+s) \sum_{q=1}^{\infty} H(q, \psi(x)+s) | < c_{41} Z^{g+1} z^{-m}; \quad (52)$$

Σ' *est étendu à tous les entiers* x *tels que* $|x| \leqq Z$ *et* $\psi(x)+s \neq 0$, *tandis que* c_{41} *dépend uniquement de* m *et du choix du polynôme* $\psi(x)$.

Remarque: Dans cette proposition je considère seulement les expressions $\pm p \pm p'$, $\pm p_1{}^2 \pm p_2{}^2 \pm p'$ et $\pm p_1{}^2 \pm p_2{}^2 \pm p_1'^2 \pm p_2'^2$. Dans l'article que je publierai bientôt dans les Mathematische Annalen (Ueber Summen von Primzahlen und Primzahlquadraten), je déduis le théorème analogue pour les expressions $Kp + K'p'$; $K_1 p_1{}^2 + K_2 p_2{}^2 + K' p'$ et $K_1 p_1{}^2 + K_2 p_2{}^2 + K_1' p_1'^2 + K_2' p_2'^2$, où K, K', K_1, K_2, K_1' et K_2' sont des entiers quelconques $\neq 0$.

Démonstration: Pour tout entier x qui est en valeur absolue $\leqq Z$, on a $|\psi(x)| \leqq c_{42} Z^g$, où c_{42} dépend uniquement du choix du polynôme $\psi(x)$. Les fonctions $\Phi(t)$, definies par (32), (33) et (34), et la fonction $F(t)$ sont en vertu de (51) égales à zéro pour tout entier t qui est en valeur absolue $\geqq \frac{1}{2} + Z^g$. Pour tout entier s qui est en valeur absolue $\geqq \frac{1}{2} + (1 + c_{42}) Z^g$ on a donc

$$F(\psi(x)+s) = 0 \text{ et } \Phi(\psi(x)+s) = 0 \qquad (|x| \leqq Z),$$

de sorte que le membre de gauche de (52) est alors égal à zéro. On peut donc supposer

$$|s| < \tfrac{1}{2} + (1 + c_{42}) Z^g.$$

Appliquons maintenant la proposition 2. De cette façon l'inégalité de Cauchy-Schwarz nous apprend

$$\{ \sum_{|x|\leqq Z}{}' \; | F(\psi(x)+s) - \Phi(\psi(x)+s) \sum_{q=1}^{[g^\sigma z^\sigma]} H(q, \psi(x)+s) | \}^2$$

$$\leqq (1+2Z) \sum_{|x|\leqq Z} | F(\psi(x)+s) - \Phi(\psi(x)+s) \sum_{q=1}^{[g^\sigma z^\sigma]} H(q, \psi(x)+s) |^2$$

$$< c_{43}^2 Z^{2g+2} z^{-m};$$

dans cette démonstration c_{43}, c_{44} et c_{45} désignent des nombres conve-

nablement choisis dépendant uniquement de m et du choix du polynôme $\psi(x)$. On a par conséquent

$$\sum_{|x| \leqq Z}' \left| F(\psi(x)+s) - \Phi(\psi(x)+s) \sum_{q=1}^{[g^\sigma z^\sigma]} H(q, \psi(x)+s) \right| < c_{43} Z^{g+1} z^{-\frac{1}{2}m}. \quad (53)$$

En vertu de $\sigma \geqq m$ le lemme précédent nous apprend

$$\sum_{x \leqq Z}' \sum_{q=[g^\sigma z^\sigma]+1}^{\infty} |H(q, \psi(x)+s)| < c_{44} Z z^{-\frac{1}{2}m} \quad . \quad . \quad . \quad (54)$$

La fonction $\Phi(t)$, définie par (32), est inférieure à

$$\underset{|u \pm u' - t| \leqq \frac{1}{2}}{\int_A^B \int_{A'}^{B'}} du\, du' \leqq \int_A^B du = B - A < B \leqq \tfrac{1}{4} Z^g .$$

La fonction $\Phi(t)$, définie par (33), est inférieure à

$$\underset{|\pm u_1^2 \pm u_2^2 + u' - t| \leqq \frac{1}{2}}{\int_{A_1}^{B_1} \int_{A_2}^{B_2} \int_{A'}^{B'}} du_1\, du_2\, du' \leqq \int_{A_1}^{B_1} \int_{A_2}^{B_2} du_1\, du_2 \leqq B_1 B_2 \leqq \tfrac{1}{4} Z^g .$$

La fonction $\Phi(t)$, définie par (34), est $\leqq Z^g$; en effet, on peut supposer que dans l'expression $\pm u_1^2 \pm u_2^2 \pm u_1'^2 \pm u_2'^2$, figurant dans (34), les termes $\pm u_1'^2$ et $\pm u_2'^2$ ont le même signe (sinon on peut changer les termes), et alors la fonction $\Phi(t)$ est inférieure à

$$\int_{A_1}^{B_1} \int_{A_2}^{B_2} du_1\, du_2 \underset{|\pm(u_1'^2 + u_2'^2) - t \pm u_1^2 \pm u_2^2| \leqq \frac{1}{2}}{\int \int} du_1'\, du_2' \leqq \pi \int_{A_1}^{B_1} \int_{A_2}^{B_2} du_1\, du_2 < 4 B_1 B_2 \leqq Z^g .$$

On a donc dans tous les cas $\Phi(t) \leqq Z^g$, de sorte que (54) entraîne

$$\left| \sum_{x \leqq Z}' \Phi(\psi(x)+s) \sum_{q=[g^\sigma z^\sigma]+1}^{\infty} H(q, \psi(x)+s) \right| < c_{44} Z^{g+1} z^{-\frac{1}{2}m},$$

donc en vertu de (53)

$$\sum_{|x| \leqq Z}' \left| F(\psi(x)+s) - \Phi(\psi(x)+s) \sum_{q=1}^{\infty} H(q, \psi(x)+s) \right| < c_{45} Z^{g+1} z^{-\frac{1}{2}m}.$$

Ceci vaut pour tout nombre naturel m, d'où suit l'assertion de la proposition 3.

Remarque: Dans la formule (52) on a

$$\sum_{q=1}^{\infty} H(q, t) = 0,$$

si t ne satisfait pas à la congruence correspondante. Si par contre t satisfait à cette congruence, on a

$$\sum_{q=1}^{\infty} H(q, t) = \varrho \prod_{\substack{w > 2 \\ w \text{ premier}}} (1 + H(w, t)) > 0;$$

en examinant l'expression $\pm p \pm p'$, on a $\varrho = 2$, et $H(w, t)$ est la fonction, définie dans le lemme 18; en examinant $\pm p_1^2 \pm p_2^2 \pm p'$, on a $\varrho = 2$, et $H(w, t)$ est la fonction, définie dans le lemme 19; en examinant $\pm p_1^2 \pm p_2^2 \pm p_1'^2 \pm p_2'^2$ finalement, on a $\varrho = 8$, et $H(w, t)$ est la fonction, définie dans le lemme 20.

Démonstration: En examinant $\pm p \pm p'$, nous avons d'après les lemmes 15 et 17

$$\sum_{q=1}^{\infty} H(q, t) = (1 + H(2, t)) \prod_{\substack{w > 2 \\ w \text{ premier}}} (1 + H(w, t)) \quad . \; . \; . \quad (55)$$

Si t ne satisfait pas à la congruence correspondante, t est impair, donc d'après le lemme 18

$$1 + H(2, t) = 0.$$

Si t satisfait à la congruence correspondante, t est pair, donc d'après le lemme 18

$$1 + H(2, t) = 2 \quad \text{et} \quad 1 + H(w, t) > 0.$$

En examinant $\pm p_1^2 \pm p_2^2 \pm p'$, nous avons également (55). Si t ne satisfait pas à la congruence correspondante, t est pair, ou bien nous avons

$$\begin{aligned} t &\equiv -1 \ (mod.\ 6) \ \text{en examinant} \quad p_1^2 + p_2^2 \pm p'; \\ t &\equiv 3 \ (mod.\ 6) \ \text{en examinant} \ \pm(p_1^2 - p_2^2) \pm p'; \\ t &\equiv 1 \ (mod.\ 6) \ \text{en examinant} \ -p_1^2 - p_2^2 \pm p'. \end{aligned}$$

La remarque du lemme 19 nous fournit $1 + H(2, t) = 0$ si t est pair, et dans les trois autres cas $1 + H(3, t) = 0$. Si par contre t satisfait à la congruence correspondante, $1 + H(w, t)$ est positif pour tout nombre premier $w > 2$ d'après la remarque, ajoutée au lemme 19; en outre t est impair, donc $1 + H(2, t) = 2$.

Considérons finalement l'expression $\pm p_1^2 \pm p_2^2 \pm p_1'^2 \pm p_2'^2$; désignons par τ le nombre des termes, figurant dans cette expression et munis du signe moins. La congruence correspondante est $t + 2\tau \equiv 4$ (*mod.* 24). Le lemme 15 nous apprend

$$\sum_{q=1}^{\infty} H(q, t) = (1 + H(2, t) + H(4, t) + H(8, t)) \prod_{\substack{w > 2 \\ w \text{ premier}}} (1 + H(w, t)).$$

Si t ne satisfait pas à la congruence correspondante, on a $t + 2\tau \not\equiv 4$ (*mod.* 8) ou $t + 2\tau \not\equiv 1$ (*mod.* 3). La remarque du lemme 21 nous fournit dans le premier cas

$$1 + H(2, t) + H(4, t) + H(8, t) = 0,$$

tandis qu'on a $1 + H(3, t) = 0$ dans le deuxième cas, d'après la remarque du lemme 20.

Si par contre t satisfait à la congruence correspondante, on a

$$1 + H(2, t) + H(4, t) + H(8, t) = 8$$

d'après la remarque, ajoutée au lemme 21; en outre $1 + H(w, t) > 0$ pour tout nombre premier $w > 2$ suivant la remarque du lemme 20.

Mathematics. — *Contribution à la théorie additive des nombres.* Par J. G. VAN DER CORPUT. (Première communication).

(Communicated at the meeting of February 26, 1938.)

Presque tout nombre naturel est un nombre premier augmenté d'un carré, c'est-à-dire, ε étant positif et N suffisamment grand, le nombre des exceptions $\leqq N$ est inférieur à εN. Si g est un nombre naturel donné, presque tout nombre naturel est un nombre premier augmenté de la $g^{\text{ième}}$ puissance d'un nombre naturel; presque tout nombre naturel est un nombre premier diminué de la $g^{\text{ième}}$ puissance d'un nombre naturel et presque tout nombre naturel est en outre égal à la $g^{\text{ième}}$ puissance d'un nombre naturel diminuée d'un nombre premier.

On obtient un résultat encore plus général en considérant un polynôme quelconque non-constant $\psi(x)$ à coefficients entiers. Soit G le plus grand nombre naturel qui est un diviseur de tous les nombres $\psi(x) - \psi(0)$, où x est entier. Presque tout nombre naturel t tel que $t - \psi(0)$ soit premier avec G, est un nombre premier augmenté de $\psi(x)$, où x désigne un nombre naturel convenablement choisi. Si le terme du degré le plus élevé de $\psi(x)$ possède un coefficient positif, presque tout nombre naturel t tel que $t - \psi(0)$ soit premier avec G, peut être mis sous la forme $\psi(x) - p$, où x désigne un nombre naturel, p un nombre premier.

Quels nombres sont égaux à un nombre premier $p > 3$ augmenté du carré d'un nombre premier $p' > 3$? Le nombre p étant un multiple de 6 augmenté de ± 1 et le carré p'^2 étant un multiple de 6 augmenté de 1, seuls les multiples de 6 et les multiples de 6 augmentés de 2 entrent en considération. Inversement, presque tout multiple positif de 6, et également presque tout multiple positif de 6 augmenté de 2 peut être écrit sous la forme $p + p'^2$, où p et p' désignent des nombres premiers > 3. D'une manière analogue on obtient que presque tout multiple positif de 6, et également presque tout multiple positif de 6 augmenté de 2 est égal au carré d'un nombre premier > 3, diminué d'un nombre premier > 3. Presque tout multiple positif de 6, et également presque tout multiple positif de 6 augmenté de 4 est égal à un nombre premier > 3 diminué du carré d'un nombre premier > 3.

Chaque nombre t qui est égal à un nombre premier > 1001, augmenté de la $1000^{\text{ième}}$ puissance d'un nombre premier > 1001 a la propriété que $t - 1$ n'est divisible par aucun des nombres 2, 3, 5, 11, 41, 101 et 251; en effet, w désignant un de ces sept nombres premiers, la $1000^{\text{ième}}$ puissance d'un nombre premier > 1001 est congru à 1 *modulo* w. Inversement, presque tout nombre naturel t tel que $t - 1$ ne soit divisible

par aucun des sept nombres nommés est égal à un nombre premier >1001 augmenté de la $1000^{\text{ième}}$ puissance d'un nombre premier >1001.

Je me demande maintenant quels nombres t possèdent la forme $t = p + p'^g$, où g désigne un nombre naturel donné, p et p' des nombres premiers $> g+1$. Désignons par G le produit des nombres premiers w tels que $w-1$ soit un diviseur de g (lorsque $g = 1000$, G est donc le produit des sept nombres premiers nommés). Pour tout facteur premier w de G on a

$$p'^g = (p'^{w-1})^{\frac{g}{w-1}} \equiv 1, \text{ donc } p + p'^g \not\equiv 1 \ (mod.\ w),$$

de sorte que seuls les nombres t entrent en considération tels que $t-1$ soit premier avec G. Comme je démontrerai, presque tout nombre naturel t tel que $t-1$ soit premier avec G est égal à un nombre premier $> g+1$, augmenté de la $g^{\text{ième}}$ puissance d'un nombre premier $> g+1$. Presque tout nombre naturel t tel que $t-1$ soit premier avec G est égal à la $g^{\text{ième}}$ puissance d'un nombre premier $> g+1$, diminué d'un nombre premier $> g+1$. Presque tout nombre naturel t tel que $t+1$ soit premier avec G est égal à un nombre premier $> g+1$, diminué de la $g^{\text{iéme}}$ puissance d'un nombre premier $> g+1$.

Considérons encore un polynôme non-constant $\psi(x)$ à coefficients entiers. Désignons par E l'ensemble des nombres naturels t tels qu'à tout nombre premier w corresponde au moins un nombre entier y non divisible par w satisfaisant à la condition que $t - \psi(y)$ ne soit pas divisible par w. Presque tout nombre de E est égal à un nombre premier augmenté de $\psi(p')$, où p' désigne un nombre premier. Si le terme du degré le plus élevé de $\psi(x)$ possède un coefficient positif, presque tout nombre de E est égal à $\psi(p') - p$, où p et p' sont des nombres premiers.

Finalement je mentionne un résultat d'un caractère un peu différent: presque tout multiple positif de 24 augmenté de 3 est la somme des carrés de trois nombres premiers.

Tous ces résultats et encore beaucoup d'autres résultent du théorème suivant. Il est vrai que le texte de cette proposition ressemble beaucoup à celui d'un autre théorème que j'ai publié autre part [1]), mais les applications sont tout autres.

Je considère une suite dénombrable de nombres positifs

$$\gamma_1, \gamma_2, \ldots \qquad \ldots\ldots\ldots\ldots \qquad (\gamma)$$

et une suite de nombre naturels

$$\eta_1, \eta_2, \ldots \qquad \ldots\ldots\ldots\ldots \qquad (\eta)$$

Désignons par V, V' et T trois intervalles, dont chacun contient au

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[1]) Sur l'hypothèse de GOLDBACH pour presque tous les nombres pairs, Acta Arithmetica, **2**, 266—290 (1937), voir p. 268 et 269.

Sur deux, trois ou quatre nombres premiers, Première communication, Proc. Royal Netherlands Acad. Amsterdam, **40**, 846—850 (1937), voir p. 849 et 850.

moins un entier. Les sommes $\sum_v$, $\sum_{v'}$ et $\sum_t$ sont étendues à tous les entiers, appartenant respectivement à V, V' en T, tandis que $\sum_{v+v'=t}$ est étendu à toutes les paires d'entiers v et v', telles que v appartienne à V, que v' appartienne à V' et que $v+v'$ soit égal à un nombre donné t.

Proposition 1: *Conditions.* 1. *Le nombre N est $\geqq 3$ et nous désignons $\log N$ par n. Chacun des intervalles V et V' contient N entiers tout au plus. Γ et Γ' désignant des nombres positifs. Pour tout entier v de V et tout entier v' de V' je définis les fonctions $r(v)$, $\varrho(v)$, $r'(v')$ et $\varrho'(v')$, telles que $\varrho'(v')$ soit en valeur absolue $\leqq \Gamma'$,*

$$\sum_v |r(v)|^2 \leqq \Gamma^2 N, \quad \sum_v |\varrho(v)|^2 \leqq \Gamma^2 N, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\sum_{v'} |r'(v')| \leqq \Gamma' N \text{ et } \sum_{v' \text{ et } v'+1 \text{ dans } V'} |\varrho'(v'+1) - \varrho'(v')| \leqq \Gamma'. \quad . \quad (2)$$

2. *Le nombre l est $\geqq 0$. A toute fraction irréductible $\frac{a}{q} \geqq 0$ et < 1 correspondent deux nombres $\lambda\left(\frac{a}{q}\right)$ et $\lambda'\left(\frac{a}{q}\right)$ tels que les inégalités*

$$\left|\lambda\left(\frac{a}{q}\right)\right| \leqq \gamma_1 q^l, \qquad \left|\lambda'\left(\frac{a}{q}\right)\right| \leqq \gamma_1 q^l, \quad . \quad . \quad . \quad . \quad (3)$$

$$\left|\sum_{v \leqq y} r(v) e^{\frac{2\pi i a v}{q}} - \lambda\left(\frac{a}{q}\right) \sum_{v \leqq y} \varrho(v)\right| \leqq \gamma_m \Gamma N q^l n^{-m} \quad . \quad . \quad (4)$$

et

$$\left|\sum_{v' \leqq y} r'(v') e^{\frac{2\pi i a v'}{q}} - \lambda'\left(\frac{a}{q}\right) \sum_{v' \leqq y} \varrho'(v')\right| \leqq \gamma_m \Gamma' N q^l n^{-m} \quad . \quad (5)$$

soient vérifiées pour tout nombre naturel m et pour toute valeur réelle de y.

3. *Pour tout nombre naturel m et pour tout nombre réel α tels que l'intervalle fermé $(\alpha - N^{-1} n^{\eta_m}, \ \alpha + N^{-1} n^{\eta_m})$ ne contienne aucune fraction à dénominateur positif $\leqq n^{\eta_m}$, on a*

$$\left|\sum_{v'} r'(v') e^{2\pi i \alpha v'}\right| \leqq \gamma_m \Gamma' N n^{-m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Sous ces conditions à tout nombre naturel m correspondent un nombre naturel $\sigma \geqq m$, ne dépendant que de m et de la suite (η), et aussi un nombre positif c_1, dépendant uniquement de m, de l et des suites (γ) et (η), tels que

$$\sum_t \left| L(t) - \Lambda(t) \sum_{q=1}^{[n^\sigma]} H(q,t) \right|^2 \leqq c_1 \Gamma^2 \Gamma'^2 N^3 n^{-m}, \quad . \quad . \quad . \quad (7)$$

où

$$L(t) = \sum_{v+v'=t} r(v) r'(v'), \quad \Lambda(t) = \sum_{v+v'=t} \varrho(v) \varrho'(v') \quad . \quad . \quad . \quad . \quad (8)$$

et

$$H(q,t)=\sum_{\substack{a=0\\(a,q)=1}}^{q-1}\lambda\left(\frac{a}{q}\right)\lambda'\left(\frac{a}{q}\right)e^{-\frac{2\pi iat}{q}}; \quad . \; . \; . \; . \; . \; . \; . \qquad (9)$$

(a, q) désigne le plus grand commun diviseur de a et q.

Pour la démonstration de ce théorème j'ai besoin de trois propositions auxiliaires [1]).

Lemme 1: *$\sum_h$ et $\sum_k$ étant étendus à un nombre fini de termes, on a*

$$\int_0^1\Big|\sum_h a_h e^{2\pi i h\alpha}\Big|\cdot\Big|\sum_k b_k e^{2\pi i k\alpha}\Big|\,d\alpha\leqq\sqrt{\sum_h|a_h|^2\cdot\sum_k|b_k|^2}.$$

Lemme 2: *Désignons par N un nombre $\geqq 3$, par V un intervalle contenant N entiers tout au plus. Si $\sum\limits_{v\leqq y} g(v)$, étendu à tous les entiers $v\leqq y$ de V est, pour toute valeur réelle de y, en valeur absolue $\leqq\varrho$, où ϱ est indépendant de y, on a pour tout nombre réel α*

$$\Big|\sum_v g(v)\,e^{2\pi i\alpha v}\Big|\leqq 3\varrho\,(1+N\,|\sin\pi\alpha|).$$

Lemme 3: *Si $\varrho'(v')$ est déterminé pour tout entier v' d'un intervalle V', de telle façon que*

$$|\varrho'(v')|\leqq\Gamma' \text{ et } \sum_{v' \text{ et } v'+1 \text{ dans } V'}|\varrho'(v'+1)-\varrho'(v')|\leqq\Gamma',$$

on a pour tout nombre réel non-entier α

$$\Big|\sum_{v'}\varrho'(v')\,e^{2\pi i\alpha v'}\Big|\leqq\tfrac{3}{2}\,\Gamma'\,|\sin\pi\alpha|^{-1}.$$

Démonstration: Le produit

$$(1-e^{-2\pi i\alpha})\sum_{v'}\varrho'(v')e^{2\pi i\alpha v'}=\sum_{v' \text{ dans } V'}\varrho'(v')e^{2\pi i\alpha v'}-\sum_{v'+1 \text{ dans } V'}\varrho'(v'+1)e^{2\pi i\alpha v'}$$

est en valeur absolue

$$\leqq\Gamma'+\Gamma'+\sum_{v' \text{ et } v'+1 \text{ dans } V'}|\varrho'(v'+1)-\varrho'(v')|\leqq 3\,\Gamma'.$$

Après ces remarques préliminaires je donnerai la démonstration du théorème. Je diviserai cette démonstration en trois parties.

Première partie de la démonstration. Désignons par m un nombre naturel arbitraire mais fixe, par $c_2, c_3, \ldots, c_{15}$ des nombres positifs, convenablement choisis dépendant uniquement de m, l et des suites (γ) et (η); désignons par σ le plus grand des deux nombres m et $\eta_{[1+\frac{1}{2}m]}$ et posons enfin $\tau=m+1+(2l+2)\sigma$.

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[1]) Pour les démonstrations des lemmes 1 et 2 on peut consulter per exemple les démonstrations des lemmes 1 et 2 de mon article: Sur l'hypothèse de GOLDBACH pour presque tous les nombres pairs. Acta Arithmetica, 2, 266—290 (1937).

Puisque chacun des intervalles V et V' contient N entiers tout au plus, il y a moins de $2N$ entiers t de la forme $t = v + v'$, où v désigne un entier de V et v' un entier de V'. $L(t)$ et $\Lambda(t)$ étant égaux à zéro pour les autres nombres t, on peut admettre sans troubler la généralité que T contient moins de $2N$ entiers.

Les inégalités (1) entraînent

$$\sum_v |r(v)| \leqq \sqrt{\sum_v 1 \,.\, \sum_v |r(v)|^2} \leqq \Gamma N \text{ et } \sum_v |\varrho(v)| \leqq \Gamma N; \quad . \quad (10)$$

$\varrho'(v')$ étant en valeur absolue $\leqq \Gamma'$, nous obtenons en utilisant la première des inégalités (2), pour les fonctions $L(t)$ et $\Lambda(t)$, déterminées par (8),

$$|L(t)| \leqq \sum_v |r(v)| \,.\, \sum_{v'} |r'(v')| \leqq \Gamma \Gamma' N^2$$

et

$$|\Lambda(t)| \leqq \Gamma' \sum_v |\varrho(v)| \leqq \Gamma \Gamma' N < \Gamma \Gamma' N^2,$$

tandis que (9) et (3) nous apprennent

$$\left| \sum_{q=1}^{[n^\sigma]} H(q, t) \right| \leqq \sum_{q=1}^{[n^\sigma]} \gamma_1^2 q^{2l+1} \leqq \gamma_1^2 n^{(2l+2)\sigma},$$

de sorte que le membre de gauche de (7) est tout au plus égal à

$$\Gamma^2 \Gamma'^2 N^4 (1 + \gamma_1^2 n^{(2l+2)\sigma})^2 \sum_t 1 < 2\, \Gamma^2 \Gamma'^2 N^5 (1 + \gamma_1^2 n^{(2l+2)\sigma})^2.$$

On peut donc admettre

$$\frac{1}{2n^{2\sigma}} > N^{-1} n^{\sigma}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

en effet, sinon N est inférieur à un nombre dépendant uniquement de m et de $\eta_{[1+\frac{1}{2}m]}$ et le théorème est alors évident.

Dans l'intervalle $0 \leqq \alpha \leqq 1$ je considère les fractions irréductibles $\frac{a}{q}$ à dénominateur positif $\leqq n^\sigma$. Ces fractions ordonnées selon la grandeur forment la suite de Farey correspondant à n^σ. Entre deux fractions consécutives $\frac{a}{q}$ et $\frac{a^*}{q^*}$ de cette suite, j'intercale la médiante $\frac{a+a^*}{q+q^*}$. Pour toute fraction $\frac{a}{q}$ appartenant à la suite et située entre 0 et 1 je désignerai par $j\left(\frac{a}{q}\right)$ le plus petit intervalle fermé contenant $\frac{a}{q}$ et borné par deux médiantes intercalées; $j(0)$ soit l'intervalle fermé $(\mu^* - 1, \lambda^*)$, où λ^* désigne la plus petite, μ^* la plus grande des médiantes intercalées.

Chacune des fractions $\frac{a}{q}$ de la suite de FAREY ayant un dénominateur positif $\leqq n^{\tau}$, chaque médiante intercalée possède un dénominateur positif $\leqq 2n^{\tau}$. Pour toute fraction $\frac{a}{q}$ de la suite de FAREY et pout toute médiante intercalée $\frac{A}{Q}$ on a donc

$$\left|\frac{a}{q}-\frac{A}{Q}\right| \geqq \frac{1}{qQ} \geqq \frac{1}{2n^{2\tau}} > N^{-1}n^{\tau} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

en vertu de (11).

Considérons une fraction quelconque $\frac{a}{q}<1$ appartenant à la suite de FAREY. Posons pour tout nombre α appartenant à $j\left(\frac{a}{q}\right)$

$$F_{\alpha}\left(\frac{a}{q}\right)=\sum_{v} r(v)\, e^{2\pi i\alpha v} \quad \text{et} \quad F'_{\alpha}\left(\frac{a}{q}\right)=\sum_{v'} r'(v')\, e^{2\pi i\alpha v'}$$

et pour les autres valeurs réelles de α

$$F_{\alpha}\left(\frac{a}{q}\right)=F'_{\alpha}\left(\frac{a}{q}\right)=0\,;$$

posons pour tout valeur réelle de α

$$\Phi_{\alpha}\left(\frac{a}{q}\right)=\lambda\left(\frac{a}{q}\right)\sum_{v}\varrho(v)\, e^{2\pi i\left(\alpha-\frac{a}{q}\right)v}$$

et

$$\Phi'_{\alpha}\left(\frac{a}{q}\right)=\lambda'\left(\frac{a}{q}\right)\sum_{v'}\varrho'(v')\, e^{2\pi i\left(\alpha-\frac{a}{q}\right)v'}.$$

Désignons par $\sum\limits_{\frac{a}{q}}$ et $\sum\limits_{\frac{a'}{q'}}$ des sommes étendues à tous les fractions $\frac{a}{q}<1$ et $\frac{a'}{q'}<1$ de la susdite suite de FAREY. Nous avons

$$\sum_{\frac{a}{q}}\int_{\frac{a}{q}-\frac{1}{2}}^{\frac{a}{q}+\frac{1}{2}} F_{\alpha}\left(\frac{a}{q}\right)F'_{\alpha}\left(\frac{a}{q}\right)e^{-2\pi i\alpha t}\,d\alpha=\sum_{\frac{a}{q}}\int_{j\left(\frac{a}{q}\right)}$$

$$=\int_{\mu^*-1}^{\mu^*}\sum_{v}\sum_{v'} r(v)\, r'(v')\, e^{2\pi i\alpha(v+v'-t)}\,d\alpha$$

$$=\sum_{v+v'=t} r(v)\, r'(v')=L(t)$$

et

$$\sum_{\frac{a}{q}} \int_{\frac{a}{q}-\frac{1}{2}}^{\frac{a}{q}+\frac{1}{2}} \Phi_\alpha\left(\frac{a}{q}\right) \Phi'_\alpha\left(\frac{a}{q}\right) e^{-2\pi i\alpha t}\, d\alpha$$

$$= \sum_{\frac{a}{q}} \lambda\left(\frac{a}{q}\right) \lambda'\left(\frac{a}{q}\right) e^{-2\pi i \frac{a}{q} t} \int_{\frac{a}{q}-\frac{1}{2}}^{\frac{a}{q}+\frac{1}{2}} \sum_{v} \sum_{v'} \varrho(v)\, \varrho'(v')\, e^{2\pi i\left(\alpha-\frac{a}{q}\right)(v+v'-t)}\, d\alpha$$

$$= \Lambda(t) \sum_{q=1}^{[n^\sigma]} H(q, t),$$

donc

$$L(t) - \Lambda(t) \sum_{q=1}^{[n^\sigma]} H(q, t) = \sum_{\frac{a}{q}} \int_{\frac{a}{q}-\frac{1}{2}}^{\frac{a}{q}+\frac{1}{2}} G_\alpha\left(\frac{a}{q}\right) e^{-2\pi i\alpha t}\, d\alpha,$$

où

$$G_\alpha\left(\frac{a}{q}\right) = F_\alpha\left(\frac{a}{q}\right) F'_\alpha\left(\frac{a}{q}\right) - \Phi_\alpha\left(\frac{a}{q}\right) \Phi'_\alpha\left(\frac{a}{q}\right).$$

Cela étant, nous obtenons

$$\sum_t \left| L(t) - \Lambda(t) \sum_{q=1}^{[n^\sigma]} H(q, t)\right|^2$$

$$= \sum_{\frac{a}{q}} \sum_{\frac{a'}{q'}} \int_{\frac{a}{q}-\frac{1}{2}}^{\frac{a}{q}+\frac{1}{2}} \int_{\frac{a'}{q'}-\frac{1}{2}}^{\frac{a'}{q'}+\frac{1}{2}} G_\alpha\left(\frac{a}{q}\right) \overline{G}_\beta\left(\frac{a'}{q'}\right) \sum_t e^{2\pi i(\beta-\alpha)t}\, d\alpha\, d\beta,$$

$\overline{w}$ désignant la valeur complexe conjuguée de w. Parce que t parcourt moins de $2N$ nombres consécutifs, on a

$$\left.\begin{aligned} &\sum_t \left| L(t) - \Lambda(t) \sum_{q=1}^{[n^\sigma]} H(q, t)\right|^2 \\ &\leqq \sum_{\frac{a}{q}} \sum_{\frac{a'}{q'}} \int_{\frac{a}{q}-\frac{1}{2}}^{\frac{a}{q}+\frac{1}{2}} \int_{\frac{a'}{q}-\frac{1}{2}}^{\frac{a'}{q'}+\frac{1}{2}} \left| G_\alpha\left(\frac{a}{q}\right) G_\beta\left(\frac{a'}{q'}\right)\right| M(\beta-\alpha)\, d\alpha\, d\beta, \end{aligned}\right\} \quad . \quad . \quad (13)$$

où

$$M(u) = Min\,(2N, |\sin \pi u|^{-1});$$

$Min\,(v, w)$ désigne le plus petit des deux nombres v et w. Par conséquent on a

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} M(u)\,du \leqq \int_{-N^{-1}}^{N^{-1}} 2N\,du + \int_{N^{-1} \leqq |u| \leqq \frac{1}{2}} |\sin \pi u|^{-1}\,du < 4 + \log N < 5n. \quad (14)$$

Dans la deuxième partie de la démonstration je démontrerai que la contribution U_1 au membre de droite de (13) des points α, β tels que

$$\left|\alpha - \frac{a}{q}\right| \geqq N^{-1} n^{\tau} \text{ et } \left|\beta - \frac{a'}{q'}\right| \geqq N^{-1} n^{\tau} \quad . \quad . \quad . \quad . \quad (15)$$

a tout au plus l'ordre de grandeur de $\Gamma^2 \Gamma'^2 N^3 n^{-m}$; dans la troisième (et dernière) partie de la démonstration je ferai voir que l'ordre de grandeur de la contribution U_2 des autres points α, β est également tout au plus égal à celui de $\Gamma^2 \Gamma'^2 N^3 n^{-m}$.

Deuxième partie de la démonstration. Dans cette partie je considère uniquement les points α, β qui satisfont à (15). On a $U_1 \leqq U_3 + 2\,U_4 + U_5$; U_3 désigne la contribution de ces points α, β à l'expression

$$\sum_{\frac{a}{q}} \sum_{\frac{a'}{q'}} \int_{\frac{a}{q}-\frac{1}{2}}^{\frac{a}{q}+\frac{1}{2}} \int_{\frac{a'}{q'}-\frac{1}{2}}^{\frac{a'}{q'}+\frac{1}{2}} \left| F_\alpha\left(\frac{a}{q}\right) F'_\alpha\left(\frac{a}{q}\right) F_\beta\left(\frac{a'}{q'}\right) F'_\beta\left(\frac{a'}{q'}\right) \right| M(\beta - \alpha)\, d\alpha\, d\beta;$$

U_4 est la contribution de ces points α, β à

$$\sum_{\frac{a}{q}} \sum_{\frac{a'}{q'}} \int_{\frac{a}{q}-\frac{1}{2}}^{\frac{a}{q}+\frac{1}{2}} \int_{\frac{a'}{q'}-\frac{1}{2}}^{\frac{a'}{q'}+\frac{1}{2}} \left| F_\alpha\left(\frac{a}{q}\right) F'_\alpha\left(\frac{a}{q}\right) \Phi_\beta\left(\frac{a'}{q'}\right) \Phi'_\beta\left(\frac{a'}{q'}\right) \right| M(\beta - \alpha)\, d\alpha\, d\beta,$$

d'où il suit, en changeant α et β, $\frac{a}{q}$ et $\frac{a'}{q'}$, que U_4 est aussi la contribution de ces points α, β à

$$\sum_{\frac{a}{q}} \sum_{\frac{a'}{q'}} \int_{\frac{a}{q}-\frac{1}{2}}^{\frac{a}{q}+\frac{1}{2}} \int_{\frac{a'}{q'}-\frac{1}{2}}^{\frac{a'}{q'}+\frac{1}{2}} \left| \Phi_\alpha\left(\frac{a}{q}\right) \Phi'_\alpha\left(\frac{a}{q}\right) F_\beta\left(\frac{a'}{q'}\right) F'_\beta\left(\frac{a'}{q'}\right) \right| M(\beta - \alpha)\, d\alpha\, d\alpha;$$

U_5 enfin est la contribution de ces points α, β à

$$\sum_{\frac{a}{q}} \sum_{\frac{a'}{q'}} \int_{\frac{a}{q}-\frac{1}{2}}^{\frac{a}{q}+\frac{1}{2}} \int_{\frac{a'}{q'}-\frac{1}{2}}^{\frac{a'}{q'}+\frac{1}{2}} \left| \Phi_\alpha\left(\frac{a}{q}\right) \Phi'_\alpha\left(\frac{a}{q}\right) \Phi_\beta\left(\frac{a'}{q'}\right) \Phi'_\beta\left(\frac{a'}{q'}\right) \right| M(\beta - \alpha)\, d\alpha\, d\beta.$$

Commençons par U_3, qui est la contribution des points nommés α, β à

$$\sum_{\frac{a}{q}} \sum_{\frac{a'}{q'}} \int\limits_{j\left(\frac{a}{q}\right)} \int\limits_{j\left(\frac{a'}{q'}\right)} \left| F_\alpha\left(\frac{a}{q}\right) F'_\alpha\left(\frac{a}{q}\right) F_\beta\left(\frac{a'}{q'}\right) F'_\beta\left(\frac{a'}{q'}\right)\right| M(\beta-\alpha)\, d\alpha\, d\beta,$$

puisque $F_\alpha\left(\frac{a}{q}\right)$ et $F_\beta\left(\frac{a'}{q'}\right)$ sont égaux à zéro en dehors de $j\left(\frac{a}{q}\right)$ et $j\left(\frac{a'}{q'}\right)$. Pour tout point α appartenant à $j\left(\frac{a}{q}\right)$ et pour toute fraction $\frac{a^\star}{q^\star}$ à dénominateur positif $q^\star \leqq n^\sigma$ on a

$$\left| \alpha - \frac{a^\star}{q^\star} \right| \geqq N^{-1} n^\tau,$$

comme on voit en utilisant (15) ou (12), selon que $\frac{a^\star}{q^\star}$ est égal à $\frac{a}{q}$ ou non. Puisque l'on a $\tau > \sigma \geqq \eta_{[1+\frac{1}{2}m]}$, l'intervalle fermé

$$(\alpha - N^{-1} n^{\eta[1+\frac{1}{2}m]}, \quad \alpha + N^{-1} n^{\eta[1+\frac{1}{2}m]})$$

ne contient aucune fraction à dénominateur positif $\leqq n^\tau$, donc aucune fraction à dénominateur positif $\leqq n^{\eta[1+\frac{1}{2}m]}$. La relation (6), appliquée avec $[1+\frac{1}{2}m]$ au lieu de m, nous apprend donc

$$\left| F'_\alpha\left(\frac{a}{q}\right)\right| \leqq c_2\, \Gamma'\, N\, n^{-[1+\frac{1}{2}m]} \leqq c_2\, \Gamma'\, N\, n^{-\frac{1}{2}-\frac{1}{2}m}.$$

De la même manière on obtient pour tout point β appartenant à $j\left(\frac{a'}{q'}\right)$ et satisfaisant à la deuxième des inégalités (15) la relation

$$\left| F'_\beta\left(\frac{a'}{q'}\right)\right| \leqq c_3\, \Gamma'\, N\, n^{-\frac{1}{2}-\frac{1}{2}m},$$

donc

$$U_3 \leqq c_2 c_3 \Gamma'^2 N^2 n^{-1-m} \sum_{\frac{a}{q}} \sum_{\frac{a'}{q'}} \int\limits_{j\left(\frac{a}{q}\right)} \int\limits_{j\left(\frac{a'}{q'}\right)} \left| F_\alpha\left(\frac{a}{q}\right) F_\beta\left(\frac{a'}{q'}\right)\right| M(\beta-\alpha)\, d\alpha\, d\beta$$

$$= c_2 c_3 \Gamma'^2 N^2 n^{-1-m} \int_0^1 \int_0^1 M(\beta-\alpha) \,.\, \left| \sum_v r(v)\, e^{2\pi i\alpha v} \right| . \left| \sum_v r(v)\, e^{2\pi i\beta v} \right| d\alpha\, d\beta$$

$$= c_2 c_3 \Gamma'^2 N^2 n^{-1-m} \int_{-\frac{1}{2}}^{\frac{1}{2}} M(u)\, du \int_0^1 \left| \sum_v r(v)\, e^{2\pi i\alpha v} \right| . \left| \sum_v r(v)\, e^{2\pi i u v} \,.\, e^{2\pi i\alpha v} \right| d\alpha$$

En utilisant le lemme 1 et les relations (14) et (1) nous obtenons

$$U_3 \leqq c_2\, c_3\, \Gamma'^2\, N^2\, n^{-1-m} \,.\, 5\, n \sum_v |\, r(v)\,|^2$$
$$< c_4\, \Gamma^2\, \Gamma'^2\, N^3\, n^{-m}.$$

U_4 est la contribution des points nommés α, β à

$$\sum_{\frac{a}{q}} \sum_{\frac{a'}{q'}} \int\limits_{j\left(\frac{a}{q}\right)} \int\limits_{\frac{a'}{q'}-\frac{1}{2}}^{\frac{a'}{q'}+\frac{1}{2}} F_\alpha\left(\frac{a}{q}\right) F'_\alpha\left(\frac{a}{q}\right) \Phi_\beta\left(\frac{a'}{q'}\right) \Phi'_\beta\left(\frac{a'}{q'}\right) \Big| M(\beta-\alpha)\, d\alpha\, d\beta.$$

L'inégalité (3) nous apprend

$$\left| \Phi'_\beta\left(\frac{a'}{q'}\right) \right| \leqq \gamma_1\, q^l \,\Big|\, \sum_{v'} \varrho'(v')\, e^{2\pi i\left(\beta-\frac{a'}{q'}\right)v'} \Big|$$
$$\leqq \frac{3}{2}\, \gamma_1\, q^l\, \Gamma' \left| \sin \pi \left(\beta - \frac{a'}{q'}\right) \right|^{-1}$$

en vertu du lemme 3. Pour tout nombre β satisfaisant à (15) on a donc

$$\left| \Phi'_\beta\left(\frac{a'}{q'}\right) \right| \leqq c_5\, \Gamma'\, n^{l\sigma}\, N\, n^{-\tau} \quad \ldots\ldots \quad (16)$$

La relation (2) étant verifiée, $F'_\alpha\left(\frac{a}{q}\right)$ est en valeur absolue $\leqq \Gamma'\, N$, de sorte que U_4 est tout au plus égal à

$$c_5\, \Gamma'^2\, N^2\, n^{l\sigma-\tau} \sum_{\frac{a}{q}} \sum_{\frac{a'}{q'}} \int\limits_{j\left(\frac{a}{q}\right)} \int\limits_{\frac{a'}{q'}-\frac{1}{2}}^{\frac{a'}{q'}+\frac{1}{2}} \left| F_\alpha\left(\frac{a}{q}\right) \Phi_\beta\left(\frac{a'}{q'}\right) \right| M(\beta-\alpha)\, d\alpha\, d\beta$$

$$\leqq \gamma_1\, c_5\, \Gamma'^2\, N^2\, n^{2l\sigma-\tau} \sum_{\frac{a'}{q'}} \int_0^1\!\!\int_0^1 M(\beta-\alpha) \Big| \sum_v r(v)\, e^{2\pi i \alpha v} \Big| . \Big| \sum_v \varrho(v)\, e^{2\pi i\left(\beta-\frac{a'}{q'}\right)v} \Big|\, d\alpha\, d\beta$$

$$= \gamma_1\, c_5\, \Gamma'^2\, N^2\, n^{2l\sigma-\tau} \sum_{\frac{a'}{q'}} \int_{-\frac{1}{2}}^{\frac{1}{2}} M(u)\, W\left(u, \frac{a'}{q'}\right) du,$$

où

$$W\left(u, \frac{a'}{q'}\right) = \int_0^1 \Big| \sum_v r(v)\, e^{2\pi i \alpha v} \Big| . \Big| \sum_v \varrho(v)\, e^{2\pi i\left(u-\frac{a'}{q'}\right)v} . e^{2\pi i \alpha v} \Big|\, d\alpha.$$

En utilisant le lemme 1 et l'inégalité (1) on trouve

$$W\left(u, \frac{a'}{q'}\right) \leqq \sqrt{\sum_v |\, r(v)\,|^2 . \sum_v |\, \varrho(v)\,|^2} \leqq \Gamma^2\, N,$$

par conséquent en vertu de (14)

$$U_4 \leqq \gamma_1 c_5 \Gamma^2 \Gamma'^2 N^3 n^{2l\sigma-\tau} \,.\, n^{2\sigma} \,.\, 5n$$
$$= c_6 \Gamma^2 \Gamma'^2 N^3 n^{(2l+2)\sigma-\tau+1} = c_6 \Gamma^2 \Gamma'^2 N^3 n^{-m}.$$

Afin de déduire une borne supérieure convenable pour U_5, j'utilise (16) et l'inégalité analogue

$$\left| \Phi'_\alpha \left(\frac{a}{q}\right) \right| \leqq c_7 \Gamma' n^{l\sigma} N n^{-\tau},$$

d'où il suit

$$U_5 \leqq c_5 c_7 \Gamma'^2 N^2 n^{2l\tau-2\tau} \sum_{\frac{a}{q}} \sum_{\frac{a'}{q'}} \int\limits_{\frac{a}{q}-\frac{1}{2}}^{\frac{a}{q}+\frac{1}{2}} \int\limits_{\frac{a'}{q'}-\frac{1}{2}}^{\frac{a'}{q'}+\frac{1}{2}} \left| \Phi_\alpha \left(\frac{a}{q}\right) \Phi_\beta \left(\frac{a'}{q'}\right) \right| M(\beta-\alpha)\, d\alpha\, d\beta$$

$$\leqq \gamma_1^2 c_5 c_7 \Gamma'^2 N^2 n^{4l\sigma-2\tau} \sum_{\frac{a}{q}} \sum_{\frac{a'}{q'}} \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} M(u)\, W^\star\left(u, \frac{a'}{q'}\right) du.$$

Dans cette formule j'ai posé

$$W^\star\left(u, \frac{a'}{q'}\right) = \int\limits_0^1 \left| \sum_v \varrho(v)\, e^{2\pi i \alpha v} \right| . \left| \sum_v \varrho(v)\, e^{2\pi i \left(u-\frac{a'}{q'}\right) v} . e^{2\pi i \alpha v} \right| d\alpha,$$

par conséquent

$$W^\star\left(u, \frac{a'}{q'}\right) \leqq \sum_v |\varrho(v)|^2 \leqq \Gamma^2 N$$

et

$$U_5 \leqq \gamma_1^2 c_5 c_7 \Gamma^2 \Gamma'^2 N^3 n^{4l\sigma-2\tau} n^{4\sigma} \,.\, 5n$$
$$= c_8 \Gamma^2 \Gamma'^2 N^3 n^{(4l+4)\sigma-2\tau+1}$$
$$= c_8 \Gamma^2 \Gamma'^2 N^3 n^{-2m-1} < c_8 \Gamma^2 \Gamma'^2 N^3 n^{-m}.$$

Ainsi nous avons démontré que la contribution U_1 est inférieure à $(c_4 + 2c_6 + c_8) \Gamma^2 \Gamma'^2 N^3 n^{-m}$.

Mathematics. — *Ueber Trivektoren.* V. Von R. WEITZENBÖCK.

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§ 13. *Die Syzygien* D_{uv}^{xy} *und* E_{uv}^{xy}.

Wir haben im § 11 in der Gleichung (98) eine Syzygie dritter Art S_3 kennen gelernt. Es war

$$D_u^x = -6\,[\hat{B}_{u\varrho,\sigma}\, d^{x\varrho\sigma} - \hat{C}^{x\varrho,\sigma}\, d_{u\varrho\sigma}] \equiv 0\ \{S_i^k\}\ ,\ \not\equiv 0\ \{\hat{B}, \hat{C}\}. \quad . \quad . \quad (108)$$

In den S_i^k ausgeschrieben, bzw. in symbolischer Darstellung haben wir

$$\tfrac{1}{6}\, D_u^x = S_\mu^\lambda\, A_\lambda^\varrho\, x_\varrho\, u^\mu - S_\mu^\lambda\, A_\sigma^\mu\, x_\lambda\, u^\sigma = (S'A)(Su')(A'x) - (SA')(S'x)(Au'). \quad (108a)$$

Eine S_3 ist linear in den $\grave{B}$, $\grave{C}$, nicht identisch Null in diesen Argumenten und verschwindet identisch in allen $S_i^k = A_i^k$, wenn die $\grave{B}$ und $\hat{C}$ in den S_i^k ausgeschrieben werden.

Wir wollen jetzt in erster Linie zeigen, dass D_u^x einen Spezialfall einer allgemeineren S_3 darstellt. Hierzu gehen wir aus von

$$\grave{B}_{uv,\varrho}\, a^{\varrho xy} = (S_u^\varrho\, c_{\varrho v} + S_v^\varrho\, c_{\varrho u})\,(ca')\, a^{xy}$$

und führen die Reduktion aus, die sich nach Gleichung (57*c*) durch den Faktor (ca') ergibt. Man erhält

$$\left.\begin{aligned} 4\,\grave{B}_{uv,\varrho}\, a^{\varrho xy} &= \{A_\varrho^x\,(v'y) + A_v^y\,(\varrho' x) - A_v^x\,(\varrho' y) - A_\varrho^y\,(v'x)\}\ S_u^\varrho + \{\ \}^\star\, S_v^\varrho = \\ &= \{(AS')(A'x)(Su')\,.\,(v'y) - (AS')(A'y)(Su')\,.\,(v'x) + \\ &\qquad\qquad + S_u^x\, A_v^y - S_u^y\, A_v^x\} + \{\ \}^\star. \end{aligned}\right\} \quad (109)$$

Hier bedeutet $\{\ \}^\star$ den Ausdruck, der aus der vorhergehenden geschweiften Klammer durch Verwechslung von u mit v entsteht. Rechter Hand haben wir jetzt zwei Terme die mit (AS') beginnen und die also nach (99*a*) durch $\grave{B}$, $\grave{C}$ und

$$J_{SA} = (S'A)\,(SA') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (110)$$

ausdrückbar sind. Wir setzen zur Abkürzung

$$\grave{B}_u^x = \hat{B}_{u\varrho,\sigma}\, d^{x\varrho\sigma}\ \ ,\ \ \grave{C}_u^x = \hat{C}^{x\varrho,\sigma}\, d_{u\varrho\sigma}, \quad . \quad . \quad . \quad . \quad . \quad (111)$$

sodass dann (108) lautet

$$D^x_u = -6\,(\dot{B}^x_u - \dot{C}^x_u) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad , \quad (112)$$

und (99*a*) wie folgt geschrieben werden kann:

$$(S'A)\,(Su')\,(A'x) \equiv \tfrac{1}{6}\,(5\,\dot{B}^x_u - \hat{C}^x_u) + \tfrac{1}{6}\,J_{SA}\,.\,(u'x) \qquad \{S^k_i\}. \quad . \quad (113)$$

Aus (109) entsteht dann

$$4\,\hat{B}_{uv,\varrho}\,a^{\varrho xy} = \tfrac{1}{6}\,J_{SA}\,.\,[(u'x)\,(v'y) - (v'x)\,(u'y)] +$$

$$+ \tfrac{1}{6}\,(5\,\dot{B}^x_u - \hat{C}^x_u)\,(v'y) - \tfrac{1}{6}\,(5\,\hat{B}^y_u - \hat{C}^y_u)\,(v'x) + \{\ \}^*$$

d.h. wir erhalten identisch in den S^k_i eine Gleichung

$$D^{xy}_{uv} \equiv 0 \;\{S^k_i\},\; \not\equiv 0 \;\{\hat{B}, \hat{C}\}$$

mit

$$\left.\begin{aligned} D^{xy}_{uv} = 24\,\dot{B}_{uv,\varrho}\,a^{\varrho xy} - (5\,\dot{B}^x_u - \dot{C}^x_u)\,.\,(v'y) - (5\,\hat{B}^x_v - \hat{C}^x_v)\,.\,(u'y) + \\ + (5\,\dot{B}^y_u - \dot{C}^y_u)\,.\,(v'x) + (5\,\hat{B}^y_v - \hat{C}^y_v)\,.\,(u'x). \end{aligned}\right\} \quad (114)$$

Dieser Tensor D^{xy}_{uv} ist. wie diese Gleichung zeigt, in den xy alternierend, in den uv dagegen symmetrisch.

Nach (112) ist $\dot{B}^x_u$ mit Hilfe von D^x_u durch $\dot{C}^x_u$ oder umgekehrt ausdrückbar. Statt (114) hat man daher auch

$$\left.\begin{aligned} D^{xy}_{uv} = 24\,\dot{B}_{uv,\varrho}\,a^{\varrho xy} - 4\,[\dot{B}^x_u\,(v'y) + \dot{B}^x_v\,(u'y) - \dot{B}^y_u\,(v'x) - \dot{B}^y_v\,(u'x)] + \\ + \tfrac{1}{6}\,[D^x_u\,(v'y) + D^x_v\,(u'y) - D^y_u\,(v'x) - D^y_v\,(u'x)]. \end{aligned}\right\} \quad (115)$$

Diese Gleichung gibt also die Reduktion von $\hat{B}_{uv,\varrho}\,a^{\varrho xy}$ auf Ausdrücke $\dot{B}^x_u$ (Gleichung (111)) und Syzygien D.

Verjüngen wir (115) bezüglich y und v, so kommt nach (112) und wegen (vgl. (97*c*))

$$\hat{B}^\lambda_\lambda \equiv 0\;\{\dot{B}\},\; \dot{C}^\lambda_\lambda \equiv 0\;\{\dot{C}\};\; D^\lambda_\lambda \equiv 0\;\{\hat{B}, \dot{C}\}: \quad . \quad . \quad . \quad (116)$$

$$D^{x\lambda}_{u\lambda} = D^x_u\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

d.h. der Tensor (108) entsteht aus (114) durch Verjüngung. Im Gegensatz zu (117) haben wir bei Verjüngung nach x und u:

$$D^{\lambda y}_{\lambda v} = -D^y_v\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (118)$$

Nochmalige Verjüngung gibt schliesslich

$$D^{\lambda\mu}_{\lambda\mu} \equiv 0 \; \{\hat{B}\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (119)$$

Dual zu D^{xy}_{uv} ist der Tensor oder die Komitante

$$\left.\begin{aligned} E^{xy}_{uv} = 24\,\hat{C}^{xy,\varrho}\,d_{\varrho uv} - (5\,\hat{C}^x_u - \hat{B}^x_u)(v'y) - (5\,\hat{C}^y_u - \hat{B}^y_u)(v'x) + \\ + (5\,\hat{C}^x_v - \hat{B}^x_v)(u'y) + (5\,\hat{C}^y_v - \hat{B}^y_v)(u'x). \end{aligned}\right\} \quad . \quad . \quad . \quad (120)$$

Hier haben wir Symmetrie in xy, Alternieren in uv. Analog zu (115) ist auch

$$\left.\begin{aligned} E^{xy}_{uv} = 24\,\dot{C}^{xy,\varrho}\,d_{\varrho uv} - 4\,[\dot{C}^x_u(v'y) + \dot{C}^y_u(v'x) - \dot{C}^x_v(u'y) - \dot{C}^y_v(u'x)] - \\ - \tfrac{1}{6}\,[D^x_u(v'y) + D^y_u(v'x) - D^x_v(u'y) - D^y_v(u'x)] \end{aligned}\right\} \quad . \, (121)$$

und ebenso ergeben sich Formeln, die zu (116) bis (119) analog sind und wovon wir nur

$$E^{x\lambda}_{u\lambda} = E^x_u = -D^x_u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (121a)$$

anführen.

§ 14. *Die Syzygien* F^{xy}_{uv}.

Wir gehen jetzt aus von

$$\hat{B}_{\varrho u,\,v}\,d^{\varrho xy} = S^\sigma_\varrho\,c_{\sigma uv}\,d^{\varrho xy} + S^\sigma_u\,c_{\sigma\varrho v}\,d^{\varrho xy};$$

hier enthält der zweite Term rechts den Faktor $c_\varrho\,d^\varrho = (c\,d')$ und kann also nach (57*c*) reduziert werden. Dies gibt

$$4\,\hat{B}_{\varrho u,\,v}\,d^{\varrho xy} = 4\,S^\sigma_\varrho\,c_{\sigma uv}\,d^{\varrho xy} + (S'A)(Su')(A'y)\,.\,(v'x) - (S'A)(Su')(A'x)\,.\,(v'y) + \\ + S^y_u\,A^x_v - S^x_u\,A^y_v\,.$$

Drücken wir hier die Glieder mit $(S'A)$ nach (113) aus, so entsteht

$$\left.\begin{aligned} 4\,\hat{B}_{\varrho u,\,v}\,d^{\varrho xy} - \tfrac{1}{6}(5\,\hat{B}^y_u - \hat{C}^y_u)(v'x) + \tfrac{1}{6}(5\,\hat{B}^x_u - \hat{C}^x_u)(v'y) = \\ = 4\,S^\sigma_\varrho\,c_{\sigma uv}\,d^{\varrho xy} + \tfrac{1}{6}\,J_{SA}\,.\,[(v'x)(u'y) - (u'x)(v'y)] + S^y_u\,A^x_v - S^x_u\,A^y_v\,. \end{aligned}\right\} \, . \; (122)$$

Diese Gleichung gibt die Reduzibilität der Syzygie S_2, die man aus (91) erhält, wenn man A^k_i durch S^k_i ersetzt. Die linke Seite von (91) wird dann nämlich

$$(Sa')(S'b)(a'x)(a'y)(bu')(bv') = S^\sigma_\varrho\,b_{\sigma uv}\,a^{\varrho xy} = S^\sigma_\varrho\,c_{\sigma uv}\,d^{\varrho xy}.$$

Dual zu (122) erhält man, ausgehend von

$$\hat{C}^{\sigma x, y} d_{\tau u v} = S_\varrho^\sigma c^{\varrho x y} d_{\tau u v} + S_\varrho^x c^{\varrho \sigma y} d_{\varrho u v}:$$

$$\left.\begin{aligned} 4 \hat{C}^{\varrho x, y} d_{\tau u v} - \tfrac{1}{6} (5 \dot{C}_v^x - \dot{B}_v^x)(u'y) + \tfrac{1}{6} (5 \dot{C}_u^x - \dot{B}_u^x)(v'y) = \\ = 4 S_\varrho^\sigma c_{\sigma u v} d^{\varrho x y} + \tfrac{1}{6} J_{SA} \cdot [(u'y)(v'x) - (u'x)(v'y)] + S_v^x A_u^y - S_u^x A_v^y. \end{aligned}\right\} \quad (123)$$

Hier kann man den ersten Term der rechten Seite, der in (122) und (123) derselbe ist, eliminieren und bekommt:

$$\hat{B}_{\varrho u, v} d^{\varrho x y} - \hat{C}^{\varrho x, y} d_{\varrho u v} - \tfrac{1}{24} (5 \hat{B}_u^y - \hat{C}_u^y)(v'x) + \tfrac{1}{24} (5 \hat{C}_v^x - \hat{B}_v^x)(u'y) +$$
$$+ \tfrac{1}{4} (\hat{B}_u^x - \hat{C}_u^x)(v'y) = S_u^y A_v^x - S_v^x A_u^y.$$

Hier ist die rechte Seite identisch Null in den $S_i^k = A_i^k$, wir erhalten also eine Syzygie dritter Art S_3:

$$\left.\begin{aligned} F_{uv}^{xy} = \dot{B}_{\varrho u, v} d^{\varrho x y} - \dot{C}^{\varrho x, y} d_{\varrho u v} - \tfrac{1}{24} (5 \hat{B}_u^y - \dot{C}_u^y)(v'x) + \\ + \tfrac{1}{24} (5 \hat{C}_v^x - \hat{B}_v^x)(u'y) + \tfrac{1}{4} (\hat{B}_u^x - \hat{C}_u^x)(v'y) \end{aligned}\right\} . \quad (124)$$

Sie ist bis auf das Zeichen zu sich selbst dual. Mit Hilfe von (112) kann sie auch wie folgt geschrieben werden:

$$\left.\begin{aligned} F_{uv}^{xy} = \dot{B}_{\varrho u, v} d^{\varrho x y} - \hat{C}^{\varrho x, y} d_{\varrho u v} - \tfrac{1}{6} \hat{B}_u^y (v'x) + \tfrac{1}{6} \hat{C}_v^x (u'y) + \\ + \tfrac{1}{144} D_u^y (v'x) + \tfrac{1}{144} D_v^y (u'y) - \tfrac{1}{24} D_u^x (v'y). \end{aligned}\right\} . \quad (124a)$$

Wir wollen jetzt beweisen, dass sowohl D_{uv}^{xy} als auch E_{uv}^{xy} reduzierbar sind auf diesen Typus F.

Hierzu bilden wir nach (124a) $F_{uv}^{xy} + F_{vu}^{xy}$, wodurch rechter Hand der zweite Term $\dot{C}^{\varrho x, y} d_{\varrho u v}$ herausfällt. Beim ersten Term hingegen entsteht

$$(\hat{B}_{\varrho u, v} + \hat{B}_{\varrho v, u}) d^{\varrho x y},$$

also nach (76), wegen der „zyklischen Symmetrie" der $\hat{B}$:

$$= - \hat{B}_{uv, \varrho} d^{\varrho x y}.$$

Daher wird:

$$D_{uv}^{xy} = - 24 (F_{uv}^{xy} + F_{vu}^{xy}) \quad . \; . \; . \; . \; . \; . \; . \quad (125)$$

Dual erhält man auf dieselbe Weise:

$$E_{uv}^{xy} = 24 (F_{uv}^{xy} + F_{uv}^{yx}). \; . \; . \; . \; . \; . \; . \quad (125a)$$

Verjüngen wir (124a) bezüglich y und v, bzw. bezüglich x und u, so entstehen

$$F^{x\mu}_{u\mu} = -\tfrac{1}{24} D^x_u = -F^x_u;\ F^{\lambda y}_{\lambda v} = \tfrac{1}{24} D^y_v = F^y_v \quad . \quad . \quad . \quad (126$$

Die Verjüngung der Gleichung (124a) bezüglich y, u und x, v dagegen ergibt:

$$F^{x\lambda}_{\lambda v} \equiv 0 \{\hat{B}, \hat{C}\},\ F^{\lambda y}_{u\lambda} \equiv 0 \{\hat{B}, \hat{C}\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (127)$$

Hieraus folgen weiters

$$F^{\lambda\mu}_{\lambda\mu} \equiv 0 \{\hat{B}, \hat{C}\} \text{ und } F^{\lambda\mu}_{\mu\lambda} \equiv 0 \{\hat{B}, \hat{C}\} \quad . \quad . \quad . \quad . \quad . \quad (128)$$

Setzt man in (124a) $u = v$ oder $x = y$, so folgen die Gleichungen

$$F^{xy}_{uu} + F^{yx}_{uu} = 0,\ F^{xx}_{uv} + F^{xx}_{vu} = 0,\ F^{xx}_{uu} = 0 \quad . \quad . \quad . \quad (129)$$

und hieraus durch Polarisieren

$$F^{xy}_{uv} + F^{xy}_{vu} + F^{yx}_{uv} + F^{yx}_{vu} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (129a)$$

§ 15. *Die Syzygien G und H.*

Nach Gleichung (101a) haben wir, identisch in den S^k_i

$$\hat{B}_{\varrho u, v} A^\varrho_w = K + (SA')(S'a)(Aw')(au')(av'), \quad . \quad . \quad . \quad (130)$$

wobei K die Komitante

$$K = (aS')(aA')(Su')(Aw')(av') \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

bedeutet. Im letzten Term von (130) ist der Faktor (SA') enthalten, wir können daher (99b) anwenden und bekommen mit Gebrauchmachung von der durch (111) und (110) eingeführten Bezeichnung:

$$\left.\begin{aligned} K &= S^\sigma_u A^\varrho_w a_{\sigma\varrho v} = \\ &= \hat{B}_{\varrho u, v} A^\varrho_w - \tfrac{1}{6}(5\hat{C}^a_w - \hat{B}^a_w)(au')(av') - \tfrac{1}{6} J_{SA} \cdot (au')(av')(aw'). \end{aligned}\right\} (132)$$

Diese Gleichung gibt, so wie nach Gleichung (101a) ausgeführt wurde, die Reduzibilität des Faktorenproduktes $(aS')(aA')$ im Gebiete der in S^k_i linearen Komitanten. Sie beweist weiters die Reduzibilität der Syzygie zweiter Art, die man aus (90a) erhält, wenn man dort $B^k_i = S^k_i$ setzt, wodurch

$$K + \tfrac{1}{6} J_{SA} \cdot (au')(av')(aw')$$

entsteht und dies ist nach (132) durch $\hat{B}$ und $\hat{C}$ auszudrücken.

Wenn wir in K S^k_i mit A^k_i vertauschen und dann subtrahieren, so entsteht

$$(aS')(aA')(Su')(Aw')(av') - (aA')(aS')(Au')(Sw')(av')$$

und dies ist identisch Null in $S_i^k = A_i^k$. Dasselbe Resultat ergibt sich, wenn man in K u' mit w' vertauscht und dann addiert. Führen wir dies in (132) aus, so fallen auch noch die Terme mit J_{SA} heraus und es ergibt sich identisch in $S_i^k = A_i^k$:

$$G_{uw,v} = \grave{B}_{\varrho u,v} A_w^\varrho + \grave{B}_{\varrho w,v} A_u^\varrho - \tfrac{1}{6}(5\grave{C}_w^a - \grave{B}_w^a)\, a_{uv} - \tfrac{1}{6}(5\grave{C}_u^a - \grave{B}_u^a)\, a_{wv} \equiv 0. \quad (133)$$

Dies ist $\not\equiv 0 \{\grave{B}, \grave{C}\}$, stellt also eine Syzygie dritter Art S_3 dar. Nach (112) können wir auch schreiben

$$\left.\begin{aligned} G_{uw,v} &= \hat{B}_{\varrho u,v} A_w^\varrho + \\ + \hat{B}_{\varrho w,v} A_u^\varrho &- \tfrac{2}{3}\hat{B}_w^a a_{uv} - \tfrac{2}{3}\hat{B}_u^a a_{wu} - \tfrac{5}{36} D_w^a a_{uv} - \tfrac{5}{36} D_u^a a_{wv}. \end{aligned}\right\} \quad (133a)$$

Wir bemerken, dass $G_{uw,v}$ symmetrisch ist bezüglich u und w.

Dual zu den G ergeben sich auf dieselbe Weise die S_3 vom Typus H. Wir haben vorerst analog zu (132)

$$\left.\begin{aligned} K' &= (a'S)(a'A)(S'x)(A'z)(a'y) = S_\sigma^x A_\varrho^z a^{\sigma\varrho y} = \\ &= \hat{C}^{\varrho x,y} A_\varrho^z - \tfrac{1}{6}(5\hat{B}_a^z - \hat{C}_a^z)\, a^{xy} - \tfrac{1}{6} J_{SA}\, a^{xyz}. \end{aligned}\right\} \quad . \quad . \quad (134)$$

Hieraus folgen dual zu (133):

$$H^{xz,y} = \hat{C}^{\varrho x,y} A_\varrho^z + \hat{C}^{\varrho z,y} A_\varrho^x - \tfrac{1}{6}(5\hat{B}_a^z - \hat{C}_a^z)\, a^{xy} - \tfrac{1}{6} J_{SA}\, a^{xyz} \quad (135)$$

und

$$\left.\begin{aligned} H^{xz,y} &= \hat{C}^{\varrho x,y} A_\varrho^z + \\ + \hat{C}^{\varrho z,y} A_\varrho^x &- \tfrac{2}{3}\hat{C}_a^z a^{xy} - \tfrac{2}{3}\hat{C}_a^x a^{zy} - \tfrac{5}{36} D_a^z a^{xy} - \tfrac{5}{36} D_a^x a^{zy}. \end{aligned}\right\} \quad (135a)$$

§ 16. *Reduktion von G und H.*

Wir beweisen jetzt, dass auch die Syzygien dritter Art vom Typus G und H auf F reduzierbar sind. Hierzu gehen wir von den in (133) auftretenden Termen

$$\tilde{B}_{u,vw} = \hat{B}_u^b b_{vw} = \hat{B}_{u\varrho,\sigma} a^{\varrho\sigma\tau} b_{\tau vw} = \hat{B}_{u\varrho,\sigma} a^{\varrho\sigma} b_{vw} (a'b) \quad . \quad . \quad (136)$$

aus und reduzieren $(a'b)$ nach $(57b)$:

$$4\tilde{B}_{u,vw} = \hat{B}_{\varrho u,w} A_v^\varrho - \hat{B}_{\varrho u,v} A_w^\varrho + \hat{B}_{uv,\varrho} A_w^\varrho - \hat{B}_{uw,\varrho} A_v^\varrho \quad . \quad . \quad (137)$$

Die rechts stehenden Ausdrücke berechnen wir aus (115) idem wir D_{uv}^{xy} mit b_{xyw} multiplizieren:

$$D_{uv}^{bb} b_w = 24\hat{B}_{uv,\varrho} A_w^\varrho - 8(\tilde{B}_{u,vw} + \tilde{B}_{v,uw}) + \tfrac{1}{3}(D_u^b b_{vw} + D_v^b b_{uw}) \quad . \quad . \quad (137a)$$

Drücken wir hier alle D nach (125) und (126) durch F aus, so entsteht:

$$\hat{B}_{uv,\varrho} A_w^\varrho = \tfrac{1}{3}(\tilde{B}_{u,vw} + \tilde{B}_{v,uw}) - (F_{uv}^{bb} b_w + F_{vu}^{bb} b_w) - \tfrac{1}{3}(F_u^b b_{vw} + F_v^b b_{uw}) \quad (138)$$

Die zwei ersten Terme der rechten Seite von (137) berechnen wir wie folgt. Wir multiplizieren (124a) mit b_{xyw}:

$$F^{bb}_{uv}\, b_w = \hat{B}_{\varrho u, v}\, A^{\varrho}_{w} - \hat{C}^{\varrho}_{w}\, d_{\varrho uv} + \tfrac{1}{6}\, \tilde{B}_{u, vw} + \tfrac{1}{6}\, \hat{C}^{\,b}_{v}\, b_{uw} - \tfrac{7}{144}\, D^{b}_{u}\, b_{vw} + \tfrac{1}{144}\, D^{b}_{v}\, b_{uw}.$$

Hier drücken wir erstens die $\hat{C}$ nach (112) durch die $\hat{B}$ und dann wieder die D durch die F aus und erhalten:

$$\left.\begin{aligned} \hat{B}_{\varrho u, v}\, A^{\varrho}_{w} = -\tfrac{1}{6}\, \tilde{B}_{u, vw} - \tfrac{1}{6}\, \tilde{B}_{v, uw} + \tilde{B}_{w, uv} + F^{bb}_{uv}\, b_w + \tfrac{7}{6}\, F^{b}_{u}\, b_{vw} + \\ + \tfrac{5}{6}\, F^{b}_{v}\, b_{wu} + 4\, F^{b}_{w}\, b_{uv}. \end{aligned}\right\} \quad (139)$$

Jetzt setzen wir (138) und (139) in die rechte Seite von (137) ein und bekommen

$$\left.\begin{aligned} 2\, \tilde{B}_{u, vw} + \tilde{B}_{v, wu} + \tilde{B}_{w, uv} = \tfrac{4}{3}\, F^{bb}_{uw}\, b_v + \tfrac{2}{3}\, F^{bb}_{wu}\, b_v - \tfrac{4}{3}\, F^{bb}_{uv}\, b_w - \tfrac{2}{3}\, F^{bb}_{vu}\, b_w - \\ - 2\, F^{b}_{u}\, b_{vw} - 3\, F^{b}_{v}\, b_{wu} - 3\, F^{b}_{w}\, b_{uv} = X. \end{aligned}\right\} \quad (140)$$

Wenn wir hier u, v und w zyklisch vertauschen, so entstehen zwei weitere Gleichungen, deren rechte Seiten wir Y und Z nennen:

$$2\, \tilde{B}_{v, wu} + \tilde{B}_{w, uv} + \tilde{B}_{u, vw} = Y.$$

$$2\, \tilde{B}_{w, uv} + \tilde{B}_{u, vw} + \tilde{B}_{v, wu} = Z.$$

Aus allen drei Gleichungen lassen sich dann die in (140) links befindlichen $\tilde{B}$ berechnen. Man erhält so

$$\tilde{B}_{u, vw} = \tfrac{3}{4}\, X - \tfrac{1}{4}\, Y - \tfrac{1}{4}\, Z,$$

oder, nach Einsetzen von X, Y und Z:

$$\left.\begin{aligned} \tilde{B}_{u, vw} = \tfrac{1}{6}\, F^{bb}_{vw}\, b_u - \tfrac{1}{6}\, F^{bb}_{wv}\, b_u + \tfrac{7}{6}\, F^{bb}_{uw}\, b_v + \tfrac{5}{6}\, F^{bb}_{wu}\, b_v - \tfrac{7}{6}\, F^{bb}_{uv}\, b_w - \\ - \tfrac{5}{6}\, F^{bb}_{vu}\, b_w - F^{b}_{v}\, b_{wu} - F^{b}_{w}\, b_{uv}. \end{aligned}\right\} \quad (141)$$

Damit sind die $\tilde{B}$ auf F zurückgeführt. Aus (138) wird dann

$$\hat{B}_{uv, \varrho}\, A^{\varrho}_{w} = \tfrac{4}{9}\, F^{bb}_{vw}\, b_u + \tfrac{2}{9}\, F^{bb}_{wv}\, b_u + \tfrac{4}{9}\, F^{bb}_{uw}\, b_v + \tfrac{2}{9}\, F^{bb}_{wu}\, b_v - \tfrac{5}{3}\, F^{bb}_{uv}\, b_w - \tfrac{5}{3}\, F^{bb}_{vu}\, b_w \quad (142)$$

und aus (139) erhalten wir:

$$\left.\begin{aligned} \hat{B}_{\varrho u, v}\, A^{\varrho}_{w} = \tfrac{11}{18}\, F^{bb}_{vw}\, b_u + \tfrac{19}{18}\, F^{bb}_{wv}\, b_u - \tfrac{19}{18}\, F^{bb}_{uw}\, b_v - \tfrac{23}{18}\, F^{bb}_{wu}\, b_v + \\ + \tfrac{3}{2}\, F^{bb}_{uv}\, b_w + \tfrac{1}{6}\, F^{bb}_{vu}\, b_w + 4\, F^{b}_{w}\, b_{uv}. \end{aligned}\right\} \quad (143)$$

Jetzt können wir die ersten vier Terme der rechten Seite von (133a) durch F ausdrücken, ebenso die beiden letzten vermöge (126). Es wird

$$3\, G_{uw, v} = (F^{bb}_{vw}\, b_u + F^{bb}_{vu}\, b_w) + 5\, (F^{bb}_{wv}\, b_u + F^{bb}_{uv}\, b_w) - (F^{bb}_{uw}\, b_v + F^{bb}_{wu}\, b_v), \quad (144)$$

d.h. also, dass $G = 0$ eine reduzible Syzygie dritter Art darstellt.

Aus (144) folgt durch zyklische Vertauschung von u, w und v, wenn $\sum\limits_0$ zyklische Vertauschung und Addieren andeutet

$$\tfrac{3}{5} \sum_0 G_{uv,w} = \sum_0 (F^{bb}_{vw} b_u + F^{bb}_{wv} b_u), \quad . \quad . \quad . \quad . \quad . \quad . \quad (145)$$

also nach (125) auch:

$$-\tfrac{1}{40} \sum_0 G_{uv,w} = \sum_0 D^{bb}_{vw} b_u . . \quad . \quad . \quad . \quad . \quad . \quad . \quad (145a)$$

Da nun nach (136)

$$\sum_0 \tilde{B}_{u,vw} = 0 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (146)$$

gilt, kann man nach (137*a*) statt (145a) auch schreiben

$$\sum_0 G_{uv,w} = -24 . 40 . \sum_0 \hat{B}_{uv,\varrho} A^{\varrho}_{w} . \quad . \quad . \quad . \quad . \quad (147)$$

Geht man dual zu (136) aus von

$$\tilde{C}^{x,yz} = \hat{C}^{x}_{b} b^{yz} = \hat{C}^{x\varrho,\tau} a_{\varrho\sigma\tau} b^{\tau yz}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (148)$$

so ergeben sich analoge Formeln. Hierbei tritt C an die Stelle von B, E an die von D, nach (125*a*) also $-F$ an die Stelle von $+F$. So wird z.B. dual zu (142) und (143):

$$\hat{C}^{xy,\varrho} A^{z}_{\varrho} = -\tfrac{4}{9} F^{yz}_{bb} b^x - \tfrac{2}{3} F^{zy}_{bb} b^x - \tfrac{4}{9} F^{xz}_{bb} b^y - \tfrac{2}{9} F^{zx}_{bb} b^y + \tfrac{5}{3} F^{xy}_{bb} b^z + \tfrac{5}{3} F^{yx}_{bb} b^z \quad (149)$$

$$\left.\begin{aligned} \hat{C}^{\varrho x,y} A^{z}_{\varrho} &= -\tfrac{11}{18} F^{yz}_{bb} b^x - \tfrac{19}{18} F^{zy}_{bb} b^x + \tfrac{19}{18} F^{xz}_{bb} b^y + \\ &+ \tfrac{23}{18} F^{zx}_{bb} b^y - \tfrac{3}{2} F^{xy}_{bb} b^z - \tfrac{1}{6} F^{yx}_{bb} b^z - 4 F^{z}_{b} b^{xy} . \end{aligned}\right\} \quad . \quad . \quad . \quad (150)$$

Hieraus findet man dann für die Syzygie H:

$$3 H^{xz,y} = -(F^{yz}_{bb} b^x + F^{yx}_{bb} b^z) - 5 (F^{zy}_{bb} b^x + F^{xy}_{bb} b^z) + (F^{xz}_{bb} b^y + F^{zx}_{bb} b^y) \quad (151)$$

und analoge Formeln dual zu (145) bis (147).

History of Science. — *Eine unbekannte Selbstbiographie von* MARTINUS VAN MARUM (1750—1837). Von ERNST COHEN.

(Communicated at the meeting of February 26, 1938.)

Es war eine von allen hochgeschätzte Festgabe, welche die Firma JOH. ENSCHEDÉ & ZONEN in Harlem den Teilnehmern der 68. Generalversammlung der „Nederlandsche Chemische Vereeniging" am 21. Juli 1931 darbot, als sie dieselben mit einem Exemplar einer Neuausgabe der „Schets der Leere van LAVOISIER" von MARTINUS VAN MARUM beschenkte und ihnen damit das Lebenswerk dieses grossen Naturforschers in das Gedächtnis zurückrief.

In den zusammenfassenden Werken, welche der Geschichte der Naturwissenschaften gewidmet sind, wird man dessen Namen vergeblich suchen.

Zwar führte H. P. M. VAN DER HORN VAN DEN BOS (1848—1913) die grosse Bedeutung von VAN MARUM seinen Fachgenossen in zwei vorzüglichen Schriften [1]) klar vor Augen, diese Abhandlungen wurden indes der Weltlitteratur nicht einverleibt. Dies ist wohl eine Folge der Tatsache, dass sie (als Antworten auf Preisaufgaben) zu holländisch in den nur wenig verbreiteten Publikationen holländischer Vereine erschienen.

Auch der von J. BOSSCHA 1887 gelegentlich des „Eerste Nederlandsch Natuur- en Geneeskundig Congres in Amsterdam" gehaltenen Rede, welche später auch in französischer Sprache in den „Archives du Musée TEYLER" zum Abdruck gelangte [2]), fiel das nämliche Los zu Teil.

Weder VAN DEN BOS noch BOSSCHA brachte uns Näheres über VAN MARUMs Lebensgang.

Gelegentlich meiner Studien über diesen hochverdienten Forscher, welche ich bereits vor etwa zehn Jahren anfing, versuchte ich näheren Daten über dessen Persönlichkeit auf die Spur zu kommen. Archivuntersuchungen ergaben, dass er nicht in Groningen das Licht der Welt erblickte, wie u.A. auch POGGENDORFF [3]) mitteilt, sondern in Delft, aus-

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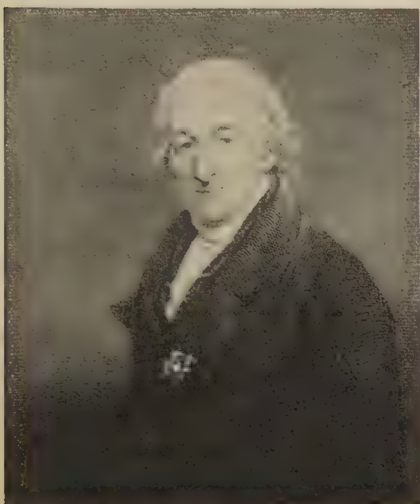
[1]) De „Nederlandsche Scheikundigen van het laatst der vorige eeuw". Utrecht (1881). Prijsvraag, uitgeschreven door het Provinciaal Utrechtsch Genootschap voor Kunsten en Wetenschappen, und „Het aandeel, dat de scheikundigen in Frankrijk, Engeland, Duitschland en Noord- en Zuid-Nederland hebben gehad in het algemeen tot erkenning brengen van het Systeem van LAVOISIER". Prijsvraag, uitgeschreven door het Genootschap ter bevordering der Natuur-, Genees- en Heelkunde, Amsterdam (1895).

[2]) Archives du Musée TEYLER, Série II, T. VI, Cinquième part.

[3]) Biographisch-Literarisches Handwörterbuch zur Geschichte der exacten Wissenschaften. Leipzig, Bd 2, 68 (1863).

serdem aber, dass einige Quellen, auf welche ich meine Hoffnung gesetzt hatte, verloren gegangen sind. Ich wies bereits in meiner Rede „*Tua res agitur*", welche ich in der Generalversammlung [1]) der Königlichen Niederländischen Akademie der Wissenschaften zu Amsterdam am 12. April 1937 hielt, darauf hin, dass einige Archivalien, welche, wie der Inventar [2]) des Archivs der „Hollandsche Maatschappij der Wetenschappen" in Harlem berichtet, sich auf van MARUMs Leben beziehen, und welche wahrscheinlich Unica sind, aus jenem Archiv verschwunden sind.



MARTINUS VAN MARUM (1750—1837).

Es war mir eine grosse Ueberraschung, als vor einiger Zeit mein Freund ERNST H. KRELAGE, der bekannte Harlemmer Botaniker, dem ich obiges mitteilte, mir erzählte, dass er im Besitze eines Büchleins sei, welches von VAN MARUM für seine Freunde geschrieben war und als Einleitung dessen Selbstbiographie enthielt. Diese Schrift, welche vier Druckbogen umfasst, trägt den Titel: „Catalogue des Plantes cultivées au printems 1810, dans le jardin de M. VAN MARUM à Harlem". Das Titelblatt besagt folgendes: „Imprimé au frais de l'auteur pour servir à ses correspondans botanistes, et faciliter les echanges reciproques". Das betreffende Exemplar enthält eine Widmung von VAN MARUM: „à Mr. G. VROLIK, Professeur de Botanique etc. etc. à Amsterdam, février 1811. No. 11". Offenbar sind die von VAN MARUM verschenkten Exemplare numeriert. In einem an VROLIK gerichteten Briefe (vom 27. Februar 1811), welcher in das genannte Exemplar eingeklebt ist, heisst es (in deutscher Uebersetzung): „Sehr geehrter Freund, Ich habe für meine botanischen Korrespondenten einen Katalog drucken lassen, von welchem ich Ihnen beigehend ein Exemplar verehre. In einigen Exemplaren, für meine speziellen Freunde, zu welchen ich Sie rechnenzudürfen hoffe, bestimmt sind, habe ich die wenigen Seiten einschalten lassen, welche nach dem Titel folgen......"

Der hier vorliegende Katalog ist offenbar der, auf welchen VAN MARUM in seinem Briefe (Juni 1819) an HAWORTH [3]) anspielt, wo es heisst: „Ich

[1]) Jaarboek der Koninklijke Nederlandsche Akademie van Wetenschappen Amsterdam, S. 152 (1936/'37).

[2]) Zusammengestellt von P. N. VAN DOORNINCK, Haarlem, 1913. Auf Seite 25 ist eine Mappe vermerkt, deren Inhalt sich auf VAN MARUMs Lebensgang bezieht. Dieses Objekt ist nicht mehr vorhanden.

[3]) ADRIAN HARDY HAWORTH (1767—1833), Entomologe und Botaniker, welcher

lege dieser ersten Sendung (Pflanzen) einen Katalog bei, welcher bereits 1810 gedruckt wurde und der etwa 3000 Arten enthält......", welchen Katalog A. J. A. UITEWAAL in seiner Abhandlung [1]) „Dr. MARTINUS VAN MARUM als Pflanzenliebhaber und Botaniker" erwähnt. UITEWAAL schreibt über diesen Katalog: „Später sagt VAN MARUM noch, dass er die Absicht habe ein Supplement dieses Katalogs folgenzulassen, und dass es ihm (UITEWAAL) leider nicht gelungen sei eine jener Ausgaben ausfindig zu machen".

Während ich später auf mehrere Punkte, diese Selbstbiographie betreffend, zurückzukommen gedenke, lasse ich dieselbe hier ohne weiteren Kommentar zum Abdruck bringen. (*Ich kopiere buchstäblich!*).

„J'ai été animé dès ma jeunesse d'un zèle peu commun pour la culture et l'observation des plantes. Les Leçons Botaniques élementaires de feu Mr. PIERRE CAMPER, le guide inappreciable de mes études, tant en Anatomie qu'en plusieurs parties de la Physiologie et de l'Histoire Naturelle, qui m'honora de son amitié jusqu'à sa mort, me firent connoitre premièrement la *Philosophie Botanique* de LINNÉ. J'examinois et determinois à l'age de 16 ans, avec facilité, suivant son systême sexuel, les espèces tant indigènes qu'exotiques qui s'offroient à mes yeux. Je m'appliquai surtout à l'étude des indigènes; j'en cultivai les moins communes que j'avois cherchées à quelque distance, et je formai le plan de perfectionner la Flore de ce païs. A ces recherches Botaniques je joignis l'étude de la structure interieure et de la Physiologie des plantes. Ayant fait in grand nombre d'observations et d'experiences à cet egard, je me proposois de les publier dans la dissertation, que je devois donner en 1773, pour obtenir le grade doctoral en Philosophie, à l'Université de Groningue. Mais peu après que l'impression en fut commencée, on me conseilla d'en donner seulement ce qui en pourroit être imprimé [2]) avant que le Prince Guillaume V, qu'on attendoit alors à Groningue, y fût reçu en qualité de Recteur de l'Université, afin de me faire jouir de l'honneur de defendre ma dissertation en sa présence, et de recevoir mon diplome doctoral de sa main; ce qu'on me fit regarder comme un moyen assuré d'obtenir une chaire de Botanique; science, pour la quelle je montrois tant de zèle. Cette esperance etoit d'autant plus fondée, puisquue j'étois pour lors le seul des

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zu jener Zeit in Chelsea lebte, war nicht nur ein berühmter Insektensammler, sondern zog auf seinem Landgut eine grosse Menge Fettpflanzen, welche er aus Kew erhielt. Ein Teil seiner Insektensammlung befindet sich heute in dem British Museum zu London, sein Herbarium in Oxford. Vergl. Dictionary of National Biography, 25, 246 (1891).

[1]) Succulenta, Maandblad van de Nederlandsche Vereeniging van vetplantenverzamelaars, **19**, 49 (1937), speziell daselbst S. 51.

[2]) Cette partie de mes observations et experiences fût publiée alors sous le titre: Dissertatio de motu fluidorum in plantis, experimentis et observationibus indagato. Peu de jours après; lors que j'obtins le grade en Medecine, je publiai une seconde dissertation, sous le titre: Quo usque fluidorum motus, et coeteroe quaedam animalium et planterum functiones consentiunt.

jeunes gens de ce païs, qui s'y fût appliqué. Peu de mois après que je reçus le grade susdit, la chaire de Botanique à l'Université de Groningue vint à vaquer; on me la promit effectivement, mais on m'en fit attendre le diplome pendant plusieurs mois. A la fin, après qu'on eut probablement oublié ses promesses, je fus informé dans le printems de 1774, qu'on en avoit favorisé quelqu'un, qui ne s'étoit aucunement appliqué à l'étude des plantes. Cet evenement me decouragea entièrement de continuer mes études dans cette science, et me fit mettre d'abord de côté tout ce que j'en avois destiné pour la presse. Bien loin de pouvoir poursuivre mes recherches sur la Physiologie des plantes, après un decouragement si inattendu et si penible, je desistai entierement de leur étude et de leur culture.

Je voulus faire alors d'autres recherches dans les sciences naturelles, afin de me distraire. Je choisis pour cet effet l'Electricité. L'Histoire de l'Electricité de M. PRIESTLEY, que j'avois reçuë dans ce tems là, me fit voir d'abord l'état actuel de la science. Mes tentatives pour les progrès de cette branche de Physique experimentale ne furent pas sans succès: j'obtins même par là quelque reputation, surtout après en avoir communiqué quelques resultats dans une dissertation imprimée à Groningue en 1776. M'étant établi ensuite à Harlem en 1776, pour y exercer la pratique de la Medicine, le Magistrat de la ville m'engagea bientôt à commencer des demonstrations de Physique experimentale, qui contribuoient beaucoup à me faire continuer mes recherches dans cette science. La Societé Hollandoise des Sciences m'offrit de plus en 1777 la Direction de son Cabinet d'Histoire Naturelle.

Je trouvai dans l'un et l'autre emploi, pendant plusieurs années, assez d'occupations, pour me distraire continuellement de la culture et de la Physiologie des plantes, pour les quelles je sentis neanmoins renaitre de tems en tems quelque affection. Je la recommençai enfin en 1783. Mais en 1784 je fus chargé de la Direction du Museum de TEYLER; je dus faire, pour cet établissement, les collections d'Instrumens de Physique et de Fossiles qui s'y trouvent, suivant un plan que j'avois formé, et me servir de ces instrumens pour le progrès de la Philosophie experimentale. Je profitai alors avec ardeur d'une occasion si longtems desirée de me vouer avec plus de facilité aux recherches physiques.

Je me proposai en premier de lieu de faire perfectionner plusieurs instrumens, dont l'experience m'avoit fait voir les défauts. Je commençai, en 1784, à faire construire la grande machine électrique, et de m'en servir pour des recherches nombreuses et d'une longue durée, dont on connoit les resultats. Je me livrai ensuite, depuis 1788, à étudier la nouvelle Chimie, à perfectionner et simplifier les instrumens trop compliqués, qu'on avoit employés pour les experiences fondamentales, sur les quelles elle étoit basée, et de m'en servir pour vérifier et répandre les découvertes chimiques les plus importantes. Toutes ces occupations et ces recherches, dont on voit les resultats, en partie, decrits dans les volumes publiés par la Société

Teylerienne depuis 1785—1798, me détournerent continuellement de suivre mon gout pour l'étude des plantes, quoique je le sentois s'accroitre toujours, et j'étois obligé alors de me borner à la culture de celles qui viennent en pleine terre.

Après avoir publié le dernier volume de mes recherches susdites en 1798, ayant achevé alors la collection des appareils physiques & chimiques, autant qu'il me parût possible dans ce tems là, je voulus donner aussi plus d'étendue et de relief à la collection des Fossiles, et j'en repris alors l'étude particuliere. Je les envisageai surtout comme des monumens de ce que notre globe a subi à sa surface, et des catastrophes qui y ont eu lieu. La Geologie fut alors mon étude cherie. J'en fis ici un cours de démonstrations pendant plusieurs hivers, en présence des Directeurs et des Membres de l'Institut Teylerien, et de quelques personnes invitées. Pendant les étés je fis plusieurs voyages à mes frais, tant pour étudier la situation des fossiles sur les lieux, que pour en rassembler et acquerir des pièces les plus choisies et les plus instructives, surtout celles, qui demontroient ou éclaircissoient l'un ou l'autre point de Geogenesie, pour la quelle je me proposai de complèter, autant qu'il m'étoit possible, la collection susdite. Mais ces frequents voyages pendant plusieurs étés me détournoient de plus en plus de l'étude des vegetaux. Mes absences me firent, même perdre la plus grande partie des plantes vivaces, que j'avois cultivées. L'étude des fossiles, et mes efforts pour former ici une collection en ce genre aussi instructive que distinguée, et pour la completer autant que possible, m'occupoient alors entièrement.

J'y fûs aussi animé par les expressions encourageantes de ceux, auxquels je fis part des resultats de ces recherches, ou qui entendirent mes demonstrations, et si j'avois pu alors continuer mes voyages, dont j'avois tiré le plus de fruit, tant pour acquerir les fossiles desirés que pour ma propre instruction, je m'en serois sans doute occupé encore pendant plusieurs années; j'aurois aussi probablement satisfait au désir de quelques personnes, dont j'apprecie le jugement, qui m'animoient de continuer mes recherches Geologiques, pour en communiquer finalement les resultats par l'impression. Mais cette étude m'eut peut être trop longtems detourné de celle des plantes, à la quelle je suis revenu enfin en 1804, après avoir fait l'acquisition d'un jardin, près de la ville, plus étendu que le précédent, où je demeure pendant l'été. Je consacre maintenant, pendant la belle saison, quelques heures par jour à la contemplation des plantes: j'ai repris mes observations et experiences pour en éclaircir la Physiologie, et je suis très content d'y être revenu; j'en sens le bon effet, puisque ma santé, qui ne permettoit plus des études si suivies, s'est visiblement fortifiée depuis ce tems là.

Les voyages fréquents, que je fis dans les années précédentes, m'ont procuré l'avantage d'être connu de plusieurs des premiers Botanistes de l'Europe, qui m'ont fait l'amitié de m'envoyer des graines: j'y dois

principalement le plaisir de posséder quelques plantes choisies et peu communes. Désirant d'en augmenter le nombre, j'ai fait imprimer ce catalogue des plantes, que je cultive, pour satisfaire en même tems au désir de mes Correspondans Botanistes, et faciliter des échanges reciproques. On y voit à chaque plante les citations des planches, qui les représentent le mieux, quand il en existe; ce que j'ai fait pour éviter toute erreur ou incertitude à l'egard des noms, aux quelles les envois des plantes sous des noms abusifs par quelques fleuristes, qui les cultivent pour en débiter, ont donné lieu, surtout chez ceux, qui ne savent pas distinguer les plantes par leurs caractères.

En parcourrant ce catalogue, on verra que je ne cultive pas des plantes, qui exigent une serre chaude pendant l'hyver, à l'exception d'un petit nombre, que je fais garder dans une serre chaude voisine. On remarquera de plus, que je me suis borné à un choix de plantes, sur tout de celles qu'on peut nommer plantes d'ornement, ou qui se distinguent par quelque proprieté remarquable. L'étenduë de mon jardin, ni mes moyens, ne me permettant pas de compléter ma collection, autant que je le désirerois.

J'ai suivi dans ce catalogue l'ordre que M. PERSOON a adopté dans son *Synopsis plantarum*, Paris 1805, 2 vol.: parce qu'on y trouve l'énumération la plus complette des espèces connuës. Je n'ai fait qu'une seule exception à l'égard de la Classe XVIII *Polyadelphia*, qu'il a inserée dans les autres, mais que j'ai conservée, puis qu'il me convenoit de la retenir pour l'arrangement systématique de mes plantes et de mes petits arbustes, que j'ai fait avant la publication du *Synopsis* susdit, les ayant fait placer dans des doubles seriës non interrompues, le long des sentiers du jardin, qui est arrangé dans le gout Anglois. Par cet arrangement je puis cultiver un nombre de plantes beaucoup plus grand, que j'aurois pu placer, dans la même étendue, sur des chassis à la manière des Jardin Botaniques ordinaires: parce que tous les petits sentiers entre les chassis, qui exigent tant de terrein, sont évités par là. Deplus les plantes se présentent ainsi beaucoup plus agréablement aux spectateurs, que lorsqu'il faut continuellement aller et retourner dans ces sentiers étroits pour regarder ce qu'on cultive. Le jardin ainsi arrangé fait voir d'abord toutes les plantes à ceux qui s'y proménent, et offre par leur grande varieté des coups d'oeil nombreux et fort agréables. On a déja commencé à suivre, pour des collections Botaniques, cet arrangement attrayant, dont j'ai donné dans ce païs le premier exemple: j'ignore si on l'avoit adopté ailleurs.

Si l'essai, que j'ai fait d'unir l'agréable au scientifique d'un Jardin Botanique, composé principalement de plantes d'ornement, pouvoit encourager dans ma patrie, la culture et l'étude des plantes, qui y a commencé de revivre, après y avoir été trop longtems négligée: — si cet essai pouvoit engager de plus en plus dans ce païs les possesseurs oisifs des campagnes, ou ceux dont les circonstances leur permettent de se donner à cette étude si attrayante, et d'en tirer les plaisirs purs, dont on jouit plus abondamment

ailleurs: — enfin si ce catalogue du choix des plantes, que j'ai fait, contenant aussi les citations des ouvrages les plus utiles pour se faciliter leur connoissance, pouvoit être tant soit peu utile à ceux qui se proposent de suivre mon exemple, en cherchant dans les heures de loisir les plaisirs purs et vraiment édifiants, que la Nature vegetale (comme la Nature entière des êtres organisés) offre si abondamment à ceux qui aiment à l'étudier: — j'oserois me flatter d'avoir utilement employé aussi à cet égard les heures, que j'y ai données, ayant senti de plus, depuis longtems, la vérité du dire du très célébre et vénérable Philosophe Chrétien PRIESTLEY, dans sa préface de *l'Histoire de l'Electricité*:

A life spent in the comtemplation of the productions of Divine power, wisdom, and goodness, would be a life of devotion. The more we see of the wonderful structure of the world, and of the laws of Nature, the more clearly do we comprehend their admirable uses to make all the percipiant creation happy: a sentiment, which cannot but fill the heart with unbounded love, gratitude, and joy."

Harlem, Fevrier 1810. M. v. M.

Zum Schlusse meinen besten Dank an Frl. P. BEYDALS, stellvertretender Archivarin des Gemeinde-Archivs in Delft, für ihre freundliche Beihilfe.

Utrecht, Januari 1938. *van 't Hoff-Laboratorium.*

Physics. — *Messungen an der verbotenen P–P-Serie des Li I Spektrums im Kohlenbogen.* Von R. SIKSNA. (Mitteilung aus dem Physikalischen Institut der Universität Utrecht.) (Communicated by Prof. L. S. ORNSTEIN.)

(Communicated at the meeting of February 26, 1938.)

Zusammenfassung.

Es wird festgestellt, dass keine wahrnehmbare Verschiebung der *Li P–P*-Linien in Bogen stattfindet. Die Linien sind asymmetrisch, höhere Glieder haben grössere Asymmetrie. Mit steigender Temperatur werden die Linien schmäler, die relative Intensität von 5*P*–2*P* wächst auch mit der Temperatur. Die Intensitäten von 3*P*–2*P* und 4*P*–2*P* sind ungefähr gleich, und ihr Verhältnis verändert sich nicht merkbar bei Veränderung der Temperatur.

§ 1. *Einleitung.*

In der von ORNSTEIN und BRINKMAN (1) entwickelten Theorie des Mechanismus der Bogenentladung kommen neben anderen, diesen Mechanismus charakterisierenden Grössen auch Grössen vor, die mit der Elektronenkonzentration in der Säule des Bogens zusammenhängen.

Einige Kenntnisse über diese Elektronenkonzentration konnte man erhalten bei Untersuchung der Linienbreite von Spektrallinien im Bogen; denn diese Breite hängt neben Dopplereffekt und Stossdämpfung auch vom Starkeffekt der kombinierten Niveaus ab. Die von diesem letzten Effekt stammende Linienverbreiterung muss im Bogen von der Konzentration der elektrisch geladenen Teilchen — Elektronen und Ionen — abhängig sein. Wäre dieser Effekt genügend gross, so könnte man aus der Intensitäts-Verteilung der Linien die Verteilung des molekularelektrischen Feldes im Bogen erhalten und damit auch die Elektronenkonzentration. Eine in dieser Richtung gefasste Arbeit über die Linien der *Li*-diffusen Nebenserie und H-Balmerserie ist von G. H. REMAN (2) ausgeführt worden, wobei es sich erwies, dass die bisherigen theoretischen Anschauungen (3) über den molekularen Starkeffekt nicht imstande sind, die experimentellen Tatsachen zu erklären.

Daher schien es von Interesse zu sein, im Bogen näher Linien zu untersuchen, die besondere Abhängigkeit von elektrischem Felde zeigen und solche Eigenschaften haben die meisten verbotene Linien.

In dieser Mitteilung wird über Messungen, die an der verbotenen *P–P*-Serie des *Li–I* im Kohlenbogen ausgeführt sind, berichtet.

Die Linien der *P–P*-Serie des *Li I* treten gut im Kohlenbogen auf (4) (5), und, wie STARK (6) festgestellt hat, ist ihre Intensität stark von vor-

handenem elektrischen Felde abhängig. Ohne Feld sind diese Linien überhaupt nicht vorhanden, sodass ihre Erscheinung im Bogen ein direkter Beweis dafür ist, dass dort elektrische Felder von beträchtlichen Grössen herrschen.

§ 2. *Experimentelle Anordnung und Messmethode.*

Als Lichtquelle diente ein Bogen in Luft zwischen 8 mm Conradty Noris Kohlen, die mit einer 3.5 mm breiten Bohrung versehen waren. Die Bohrung war bei einigen Aufnahmen mit Li_2CO_3 und bei anderen mit Li_2CO_3 und KCl im Verhältnis 1 : 1 gemischt, gefüllt. Der Zusatz von KCl war notwendig, um mit Hilfe von K-Linien, deren Uebergangswahrscheinlichkeit bekannt ist (7), die im Bogen herrschende Temperatur nach dem in Utrecht ausgearbeiteten Verfahren (8), zu bestimmen. Bei den Aufnahmen mit Li_2CO_3 war es möglich, die Temperatur aus Intensitäten der nicht-verbotenen Li-Linien zu bestimmen; denn bei diesen Aufnahmen handelte es sich um Vergleich zwischen P–P und P–S-Linien. Da die P–S-Linien überwiegend intensiver als die P–P-Linien sind, war eine Hälfte des Spektrographenspaltes mit einem Abschwächer [1]) bedeckt, da sonst der Vergleich der verbotenen P–P-Linien mit den überbelichteten anderen Li-Linien nicht möglich gewesen wäre. Der Bogen wurde mit einer achromatischen Quarz-Fluorit-Linse auf dem Spalt des Hilger El. Spektrographen (Dispersion Tab. I) abgebildet und die Spektra wurden photographiert. Auf derselben Platte wurden die Spektra der Wolfram-Bandlampe bei verschiedenen Stromstärken aufgenommen. Aus der bekannten (9) Energieverteilung des kontinuierlichen Spektrums der Lampe war es möglich, nach der üblichen Utrechtschen photographisch-photometrischen Methode (10) die erhaltenen Spektra zu photometrieren. Die Photogramme wurden mit dem Apparat von WOUDA (11) in Intensitäten umgesetzt.

Der Bogen wurde mit Gleichstrom und auch gelegentlich mit Wechselstrom betrieben. Bei Wechselstromspeisung war zwischen Bogen und Spektrographenspalt ein Synchronmotor mit geschlitzter Scheibe eingesetzt, wodurch es möglich war, die Spektra des Bogens in verschiedenen Phasen und damit auch bei verschiedenen Temperaturen zu photographieren (12). Mit dieser Anordnung bei Wechselstrombetrieb sind die Expositionszeiten ungefähr 60 mal grösser, als bei direkten Gleichstromaufnahmen. Wegen der relativen Schwäche der verbotenen Li-Linien muss man daher sehr empfindliche Platten anwenden. Dadurch entstehen Schwierigkeiten bei der Auswertung der Mikrophotogramme; denn die empfindlichen Platten zeigen grobes Korn.

Die angewandten Platten für Wechselstromaufnahmen waren die Ilford Hypersensitive Panchromatic Plates, für Gleichstromaufnahmen — Ilford Special Rapid Panchromatic. Auf letzteren war das Korn viel weniger störend.

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[1]) Herrn phil. cand. H. LOBSTEIN danke ich für die Herstellung dieses Abschwächers.

Neben den Schwierigkeiten bei der Auswertung der Mikrophotogramme, die von der Körnung der Platte herrühren, treten noch andere auf, wegen der Nähe der *P–P*-Linien an die *P–D*-Linien. Die *P–P*-Linien liegen auf den Flügeln der unvermeidlich überbelichteten sehr verbreiterten Linien der diffusen Nebenserie.

Einige Schwierigkeiten rühren auch von den Linien der Elemente her, die als Verunreinigung in den Kohlen vorkommen, besonders von den Eisenlinien.

§ 3. *Experimentelle Ergebnisse.*

a. Wellenlängen. Obwohl es anfangs nicht unsere Absicht war, die Wellenlängen der untersuchten Linien zu bestimmen, erwies sich dies doch als notwendig. Anlass dazu gaben die störenden Eisenlinien. In der Gegend des vierten Gliedes der *P–P*-Serie λ 3923 Å waren mehrere *Fe*-Linien vorhanden, sodass keine Aussichten bestanden diese Linie zu messen. Doch auf späteren Aufnahmen erwies sich auch diese Gegend als photometrierbar und so entstand die Intensitätenverteilungskurve Fig. 1.

Nach STARK's Angaben (6) hat das dritte Glied der verbotenen *Li*-Serie

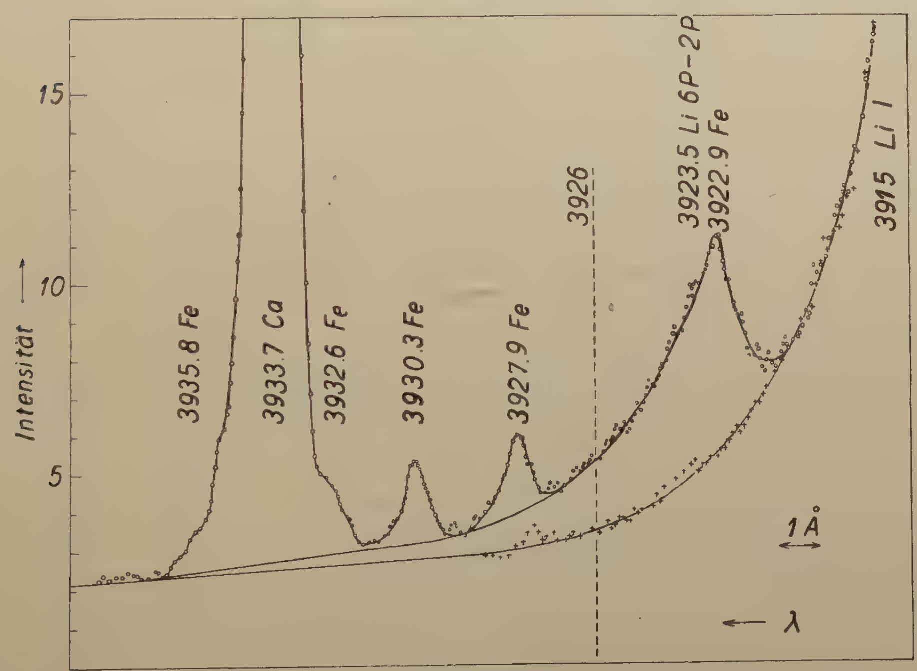


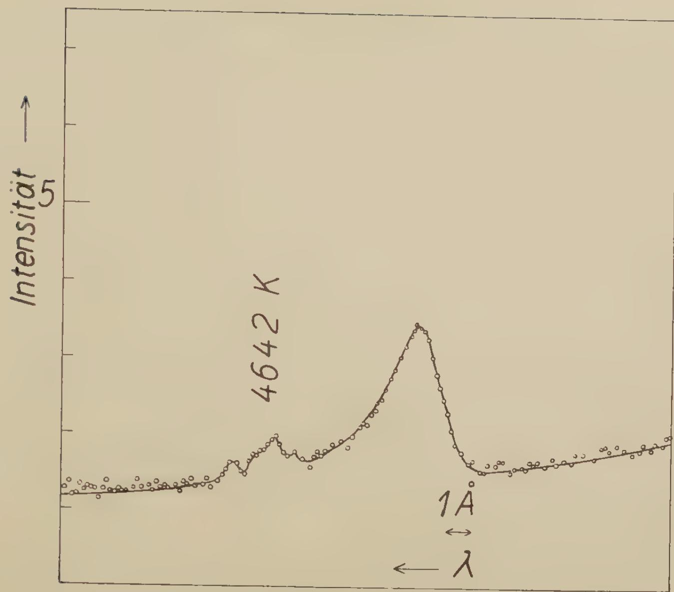
Abb. 1.
λ 3923 Å 6*P*—2*P* und Umgebung. + + + Verlauf des violetten Flügels von λ 3915 Å

λ 4147 Å in Bogen eine Verschiebung nach längeren Wellenlängen, die der Feldstärke 27000 Volt/cm entspricht. Mit Hilfe der von STARK in Kanalstrahlröhre gemessenen Werte der Verschiebung in Abhängigkeit von der Feldstärke schätzen wir die dieser Feldstärke entsprechende

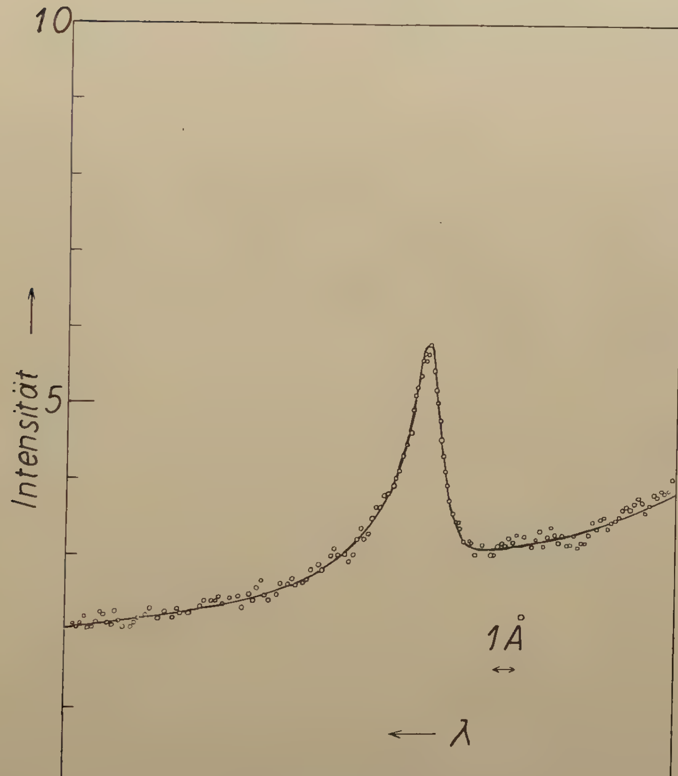
Verschiebung der Linie λ 4147 Å auf 1.7 Å. Für das vierte Glied λ 3923 Å hat STARK einen ungefähr doppelt so grossen Einfluss des elektrischen Feldes auf die Verschiebung der Linie festgestellt; (bei 80.000 Volt/cm für 4147 Å—9 Å, für 3923 Å—16 Å). Daher konnte man im Bogen für

TABELLE I. Wellenlängen der *Li I P—P*-Serie

	$3P-2P$ λ Å	Dispersion Å/mm	$4P-2P$ λ Å	Dispersion Å/mm	$5P-2P$ λ Å	Dispersion Å/mm	$6P-2P$ λ Å	Dispersion Å/mm
Berechnet	6239.7		4636.0		4146.72		3923.17	
Gemessen	6240.6 ± 0.7	44.5	4635.7 ± 0.3	20	4146.8 ± 0.3	13.6	3923.5 ± 0.5	11.6
Konen u. Hagenbach	6240.8	16.9	4636.14	16.9	4149.1	16.9	3924	16.9
Saunders					4148.3			
Stark extrapol.					4146.7	21.1	3923.2	15.4
Stark Bogen					4148.4	21.1		



λ 4636 Å 4P—2P.
Abb. 2.
T = 3700° K.



λ 4636 Å 4P—2P.
Abb. 3.
T = 5200° K.

die Linie λ 3923 Å eine Verschiebung von ungefähr 3 Å erwarten, was eine Wellenlänge von λ 3926 Å für das vierte Glied der *Li P–P*-Serie im Bogen ergibt. Diese Wellenlänge ist in Fig. 1 mit einer gestrichelten Linie eingezeichnet. Es ist sehr deutlich zu sehen, dass keine so grosse Verschiebung vorhanden ist.

Zur Bestimmung der Wellenlängen der *P–P*-Linien wurde das *Li*-Bogenspektrum mit Vergleichseisenbogenspektrum aufgenommen und aus Komparatormessungen die Wellenlängen bestimmt.

In Tabelle I sind die aus *P*-Termen (13) berechneten, die gemessenen und von anderen Autoren publizierten Zahlen für die Wellenlängen der *Li P–P*-Serie zusammengestellt. Neben diesen Zahlen ist die Dispersion der verwendeten Spektralapparate angegeben. Wie aus dieser Tabelle und Fig. 1 ersichtlich ist, ist im Bogen keine wahrnehmbare Verschiebung der *Li P–P*-Linien festzustellen.

b. Linienform. In Fig. 2 und 3 sind die drei ersten Linien der Serie, nach Umsetzung der Photogramme in Intensitäten, dargestellt. Um die Kornstörungen der Platte möglichst zu eliminieren, wurde für die Linien, welche mit den empfindlichen Platten aufgenommen sind (Fig. 2), der Durchschnittswert aus fünf ausgemessenen Intensitätskurven genommen, während Fig. 3 und ebenso Fig. 1 eine direkte Intensitätsumsetzung eines einzelnen ausphotometrierten Spektrums gibt.

Aus den Figuren ist es ersichtlich, dass die Linien asymmetrisch sind, mit geringerem Ansteigen an der roten Seite und steilerem Ablauf nach der violetten, wie dies auch schon die oben zitierten Autoren (4, 5, 6) angegeben haben. Die Asymmetrie, die durch das Verhältnis zwischen dem roten Teil der Halbwertsbreite zum violetten charakterisiert werden kann, wächst mit steigender Gliednummer. Vorläufige Zahlen darüber sind in Tabelle II angegeben. Dort befinden sich auch die direkt aus den

TABELLE II.

λ Å	Asymmetrie		Halbwertsbreite in Å gemessen		Halbwertsbreite in Å korrigiert	
	5200°	3700°	5200°	3700°	5200°	3700°
6240	1.5	1.5	4.0	4.4	3.3	3.8
4636	2.0	1.9	1.96	2.9	1.6	2.7
4147	2.1	2.2	2.0	3.4	1.8	3.3
3923	4		2.3		2.3	

Intensitätskurven gemessenen Halbwertsbreiten der Linie. Wie ersichtlich, sind die Breiten nicht so gross, wie sie im Wasserstoffbogen beobachtet sind (2). Um die wahre Halbwertsbreite und damit auch den wahren

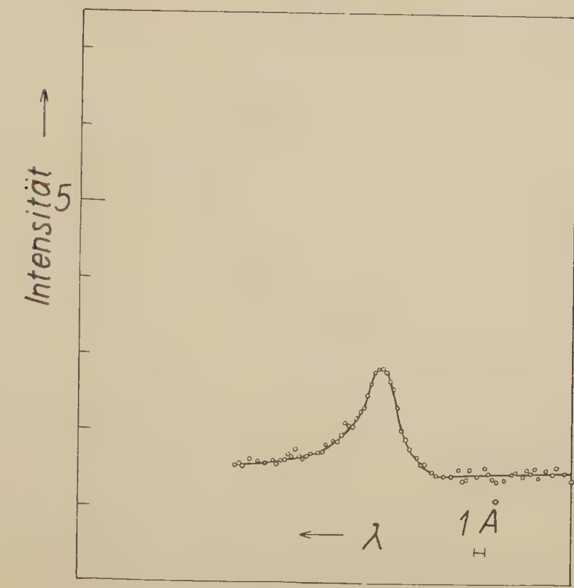
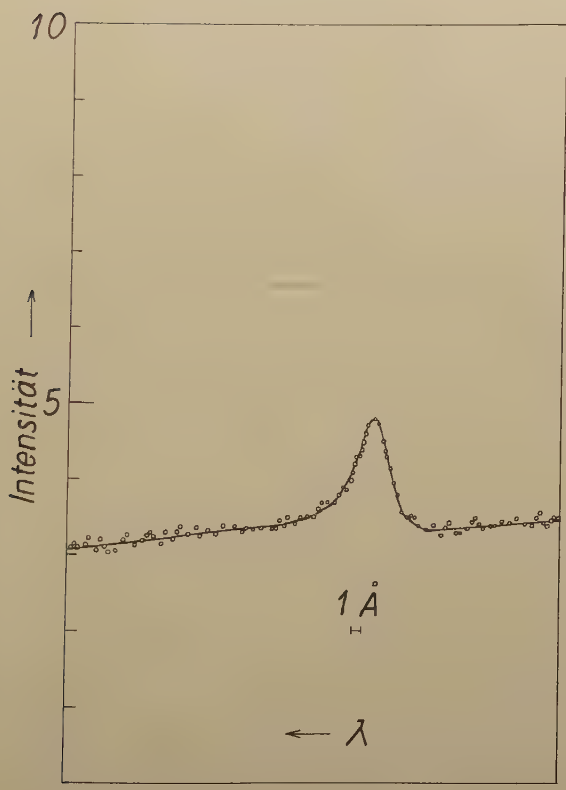
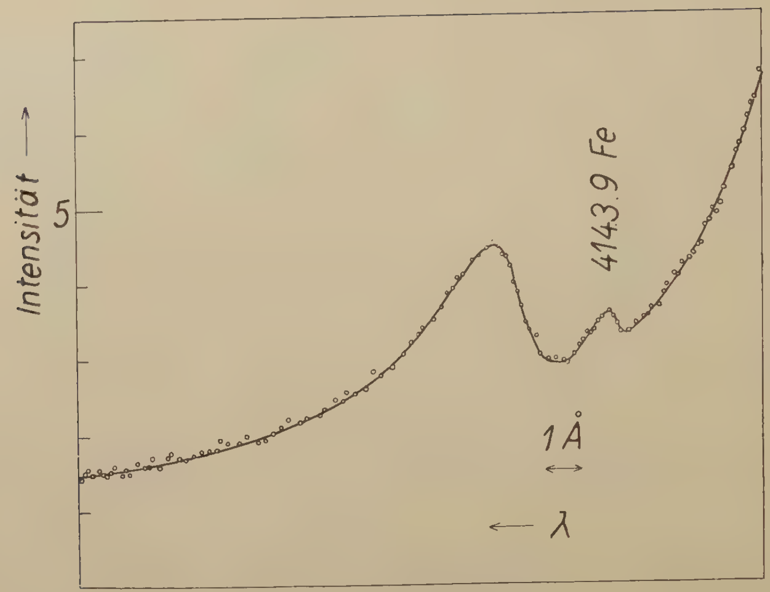


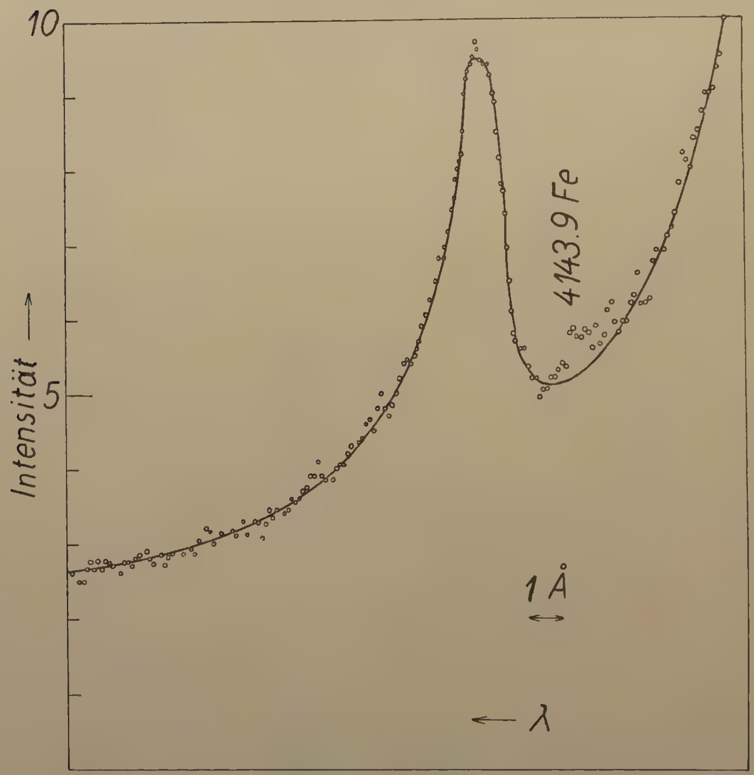
Abb. 4.
λ 6240 Å   3P—2P.
T = 3700° K.



λ 6240 Å   3P—2P.
Abb. 5.
T = 5200° K.



λ 4147 Å   5P—2P.
Abb. 6.
T = 3700° K.



λ 4147 Å   5P—2P.
Abb. 7.
T = 5200° K.

Verlauf der Intensitäts-Wellenlängenkurven der Linie zu bestimmen, ist es notwendig, die Apparatenverbreiterung zu eliminieren. Eine Untersuchung dieser Frage ist im hiesigen Laboratorium eingeleitet. Diese Messung ist für die Klärung der Frage der Linienbreite unbedingt notwendig, was aus den Zahlen der letzten Spalte der Tabelle II ersichtlich ist, wo ein ganz roher Versuch, die Halbwertsbreite durch nichtverbreiterte Linienbreite zu korrigieren, ausgeführt ist wobei die bekannte quadratische Relation benutzt wurde. Von dem zweiten Glied der Serie (wo die Apparatenbreite auf 1 Å geschätzt ist) aufwärts, nimmt die Linienbreite zu, sodass für das erste Glied eine viel kleinere Breite zu erwarten ist. Für die vorhandene Breite dieser Linie ist die geringe Dispersion und damit auch die grosse Apparatenbreite in der Gegend der Linie verantwortlich.

Die nächste merkwürdige Tatsache, die bei der Betrachtung der Linien auffällt, ist, dass die Linien bei steigender Temperatur eine kleinere Breite aufweisen. Wenn dies auch aus den angegebenen Kurven, die mit verschiedenen Stromarten und auch verschiedenen Plattensorten aufgenommen sind, nicht ganz überzeugend hervortritt, so ist diese Tatsache auch auf

TABELLE III.

Relative Intensitäten der *LiI P-P*-Serie.

	λ 6240 Å $3P-2P$		λ 4636 Å $4P-2P$		λ 4147 Å $5P-2P$		λ 4273 Å $4S-2P$	λ 4972 Å $3S-2P$	T K°
Gleichstrombogen 5.8 Amp.	1		1.1		2.05				5200
		1.05		1.16		2.16	19.7	100	
	1		0.92		1.53				4660
		1.39		1.28		2.13	17.6	100	
	1		0.97		1.53				4560
		1.26		1.22		1.92	17.5	100	
	1		1.04		1.54				4400
		1.25		1.30		1.93	17	100	
Wechselstrombogen 5.5 Amp. Phase 30°	1		0.97		1.08				3700
9.6 Amp. Phase 45°	1		1.03						

einer Platte, die mit Wechselstrom in zwei verschiedenen Phasen und damit bei zwei verschiedenen Temperaturen aufgenommen ist, festzustellen. Diese Platte wird noch endgültig bearbeitet werden.

c. Intensitäten. In Tabelle III sind die relativen intensitäten der Linien dargestellt, wozu als Einheit in einer Reihe die Intensität von λ 6240 Å dient, in anderer — λ 4972 Å ($3S$—$2P$). Die Intensitäten von $3P$—$2P$ und $4P$—$2P$ sind fast dieselben und ihr Verhältnis verändert sich nicht mit der Temperatur. Die Intensität von $5P$—$2P$ steigt mit wachsender Temperatur.

§ 4. *Folgerungen.*

Dass die Linien bei höheren Temperaturen, wo auch grössere Konzentration von elektrisch geladenen Teilchen und damit auch stärkere molekularelektrische Felder erwartet wären, schmäler werden, ist vielleicht dadurch zu erklären, dass der Intensitätsverlauf der *Li* P–P-Linien in Abhängigkeit von der Feldstärke am Anfang mit wachsender Feldstärke auch ein Ansteigen aufweist, bei höherer Feldstärke aber wieder Abfall zeigt. Genaue Aussagen darüber werden nur dann möglich sein, wenn dieser Verlauf bekannt wird.

Die verhältnissmässig grosse Abhängigkeit der Linienform von der Temperatur erlaubt, die Vermutung auszusprechen, dass die untersuchten verbotenen *Li*-Linien bei Untersuchungen der elektrischen Erscheinungen im Bogen dienen können.

Dem Herrn Direktor des Physikalischen Laboratoriums der Reichsuniversität Utrecht, Herrn Prof. Dr. L. S. ORNSTEIN möchte ich auch an dieser Stelle meinen herzlichen Dank für die freundliche Aufnahme in seinem Laboratorium, die Anregung zu dieser Arbeit und die rege Unterstützung bei ihrer Ausführung aussprechen. Auch Herrn Dr. H. BRINKMAN danke ich herzlich für viele wertvolle aufklärende Diskussionen, und ebenfalls Herrn phil. cand. J. P. A. VAN HENGSTUM und Herrn W. SIMONS, Drs. phil., die mir bei der Ausführung dieser Arbeit sehr wertvolle Hilfe leisteten und mit Rat und Tat zur Seite standen.

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Chemistry. — *The Exact Measurement of the Specific Heat of Metals at High Temperatures.* **XXX.** *The Specific Heats of Pure Iron between 25° and 1500° C.* By A. J. ZUITHOFF. (Communicated by Prof. F. M. JAEGER).

(Communicated at the meeting of February 26, 1938.)

§ 1. In 1935 Mr. G. NAESER [1]) published the results of a series of very carefully executed measurements of the mean specific heats of electrolytically prepared *iron* between 25° and 700° C. On this occasion he stated that the corresponding $\bar{c}_p$-*t*-curve of *α-iron* showed a typical wave-like form, presenting a number of successive discontinuities which are repeated with a conspicuous regularity at temperature-intervals each time extending over about 86° to 116°; — thus confirming the occurrence of similar irregularities in the physical behaviour of *α-iron* already previously observed at about the same temperatures by a number of other investigators [2]). NAESER states that the phenomena observed show a perfect reproducibility and complete reversibility on heating as well as on cooling; moreover, that they are neither due to recrystallisation, nor to flaws in his method of measurement or in the experimental device used. Although at that moment *no* rational explanation of their true nature could be given, yet the author was inclined to the supposition that in some way they should be connected with the ferromagnetic properties of *α-iron* below its CURIE-point at 760°—780° C.

Similar phenomena were, quite independently, stated in this laboratory [3]) in the case of purest ductile *titanium* and imputed to the presence of unavoidable traces of *gases*, — more especially of *oxygen*, — in the metal (loco cit., pp. 652, 653) and the gradual subsequent reductions of the several oxides, exerted by the heated metal present in excess within definite ranges of the temperature, co-inciding with their successive fields of stability. That the presence of even minute traces of such gases often may be the cause of a radical change in the behaviour of metals and, more particularly, may give rise to the manifestation of remarkable hysteresis-phenomena, has on several occasions been emphasized by different investigators [4]).

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[1]) G. NAESER, Mitt. Kaiser-Wilh. Institut f. Eisenforschung Düsseldorf, Abh. *285*, Bnd. **17**, 185 (1935).

[2]) A complete summary of the literature on the subject is given in NAESER's paper. Cf. moreover: GMELIN's *Handbuch der Anorg. Chemie*, 8e Aufl., Eisen, Teil *A*, Tief. **2**, 233.

[3]) F. M. JAEGER, E. ROSENBOHM and R. FONTEYNE, Recueil d. Trav. Chim. d. Pays-Bas, **55**, 622 (1936).

[4]) F. M. JAEGER, E. ROSENBOHM and R. FONTEYNE, loco cit., 653; J. H. DE BOER, P. CLAUSING and J. D. FAST, Recueil d. Trav. d. Chim. d. Pays-Bas, **55**, 450, 459 (1936).

In the course of an investigation still going on in this laboratory concerning the heat capacity of *iron-nickel*-alloys, — it also proved desirable once more to determine the specific heats of the pure electrolytical *iron* itself, used in these experiments, within the range of 25°—1500° C. with as great an accuracy as possible; and thus simultaneously the opportunity presented itself again to test the occurrence of the phenomena mentioned in the above. In the present paper the results of this study are communicated in detail.

§ 2. *Iron*, — the technical as well as the pure metal, — up to its melting-point shows several reversible phase-transitions, characterized by definite transition-temperatures which, under a pressure of one atmosphere, are situated at 760°—768° C., at 906° C., at 1401° C. and at 1530° C. respectively, — the latter being the melting-point of the metal. The phases successively appearing and coexistent at these temperatures are discerned as *α-*, *β-*, *γ-* and *δ-iron*; amongst them *α-*, *β-* and *δ-iron* possess *the same crystalline structure*, namely a cubic, body-centred structure (*A*-2-type) with $a_0 = 2{,}86$ A.U. (16° C.), 2,90 A.U. (800° C.), 2,93 A.U. (1425° C.) and, at present, they are considered to represent *the same* inner state of the metal, although *α-iron* is strongly ferromagnetic, whilst *β-* and *δ-iron* are feebly paramagnetic, — the ferromagnetism disappearing more or less sharply at the CURIE-point. On the other hand *γ-iron* has a deviating crystal-structure, being cubic with a face-centred lattice (*A*-1-type) and $a_0 = 3{,}63$ A.U. (1100° C.); its field of stability (906°—1401° C.) interrupts that of the body-centred cubic structure and is, therefore, inserted between that of *β-* and *δ-iron*. As to the reproducibility of the subsequent transition-points, — usually discerned as A_2, A_3 and A_4, — it may be remarked that the corresponding data, as given by different authors using the determination of different physical properties of the metallic phases, show rather considerable divergencies: in the literature the values of A_2 ($\alpha \rightleftarrows \beta$) vary between 750° and 810° C., those of A_3 ($\beta \rightleftarrows \gamma$) between 875° and 917° C. and those of A_4 ($\gamma \rightleftarrows \delta$) between 1390° and 1420° C.

These oscillations of the values for A_2, A_3 and A_4 are, — especially in the experiments with technical, i.e. carbon-containing *iron*, — partially due to differences in composition of the samples used, but, on the other hand, equally so to certain hysteresis-phenomena manifesting themselves on heating and on cooling, in the study of the dilatometrical, magnetic, electrical and mechanical changes of the metal, as well as in that of its thermal behaviour. Especially the values of the specific heats of *iron* at various temperatures, as they may be found in the literature, prove to be highly discordant and, when represented in the same graphical diagram, they appear to be scattered over it in a most unsatisfactory way.

§ 3. *Measurements of $\bar{c}_p$ of Electrolytic Iron in a Vacuum.*

The measurements of the mean specific heats of pure, electrolytically

precipitated *iron* were executed in platinum vacuum-crucibles of the type always used in this laboratory and by means of the metal block calorimeter already often described in detail. Electrolytic *iron* (from HERAEUS' *Vakuum-schmelze*) in the shape of a massive bar was turned off on the lathe and brought into the form of a cone exactly fitting within the vacuum-crucible. The weight of the *iron* enclosed in the latter under continued evacuation, was 34,121 grammes; that of the platinum crucible itself was 27,585 grammes. All measurements were made with the necessary precautions and under often repeated control of the thermocouples and the constancy of the instrument; they were executed in an arbitrary sequence, so as to be sure that no recrystallisation-phenomena would influence the tests for reproducibility of the results at each selected temperature.

At first some preliminary measurements of $\bar{c}_p$ were made at 300° C.; then, — because the values proved not at all to be constant, — the metal was several times heated to about 900° C., till the values of c_p proved to have become quite reproducible within 0,1—0,2 %.

Originally the crucible, after cooling, appeared to be somewhat inflated, evidently as a consequence of gases having been given off by the heated metal. The $\bar{c}_p$-t-curve now obtained (g/g in Fig. 1) truly showed a number of irregularities at almost the same temperature-intervals as indicated by NAESER and by BORELIUS, as may be seen from the following data:

ZUITHOFF:	NAESER:	BORELIUS: [1])
—	about 112°—198°	about 120°—200°
about 272°—285°	275°—292°	291°—301°
380°—390° 395°—404°	355°—390°	377°—392°
—	455°—487°	473°—478°
500°—525°	500°—525°	513°
602°—610°	591°—600°	564°—609°
		etc.

The temperature-intervals can only be indicated with some approximation; but on heating and subsequent cooling they appear to be sufficiently reproducible, as also was stated by NAESER.

On repeatedly heating the crucible at temperatures surpassing 950° C. and cooling to lower temperatures, the inflation of the crucible and simul-

1) G. BORELIUS and F. GUNNESON, Ann. d. Phys., (4), 67, 227 (1922); G. BORELIUS, loco cit., 243 (thermoelectrical effects).

taneously the irregularities in the curve *g/g* gradually disappear, so that finally the smooth curve *GG* is obtained which in all further experiments now remains unaltered. From these facts the conclusion can be drawn, that the said irregularities in the behaviour of α-iron between 100° and 760° C. are doubtlessly caused by and intimately connected with the presence and the final expulsion of *gases* dissolved in the metal, — most probably of *hydrogen* in this case; in the repeated heatings at 950°, — at which temperature *platinum* is already permeable for hydrogen, — the gas is gradually expelled through the wall of the crucible and a smooth c_p-t-curve is the final result, as soon as *all* hydrogen is thus removed from the crucible, so that a resorption of the gas by the metal on cooling is prevented.

That electrolytic *iron* may occasionally absorb greater quantities of *hydrogen* and that, on heating, the latter is *not* gradually given off, but *in successive jumps* at a whole series of separated temperatures, as well as the fact that this expulsion of the gas is each time accompanied by a development of heat, — has already been stated by ROBERTS–AUSTEN and by MÜLLER [1]). Exactly as in the case of the *titanium* previously mentioned, the irregularities in the $\bar{c}_p$-t-curve here observed by us as well as by NAESER, BORELIUS and other investigators, are, therefore, by no means characteristic of the metal itself, but only of ***the metal imbued with a gas.***

Indeed, by a special investigation, — in which the absorbed gases were expelled from th metal by carefully heating the latter in a vacuum and the gaseous mixture, after collection, was spectroscopically examined, — the fact was established that the gas consisted of *hydrogen* and of *nitrogen*, accompanied by traces of *oxygen*. Evidently some *air* had been absorbed

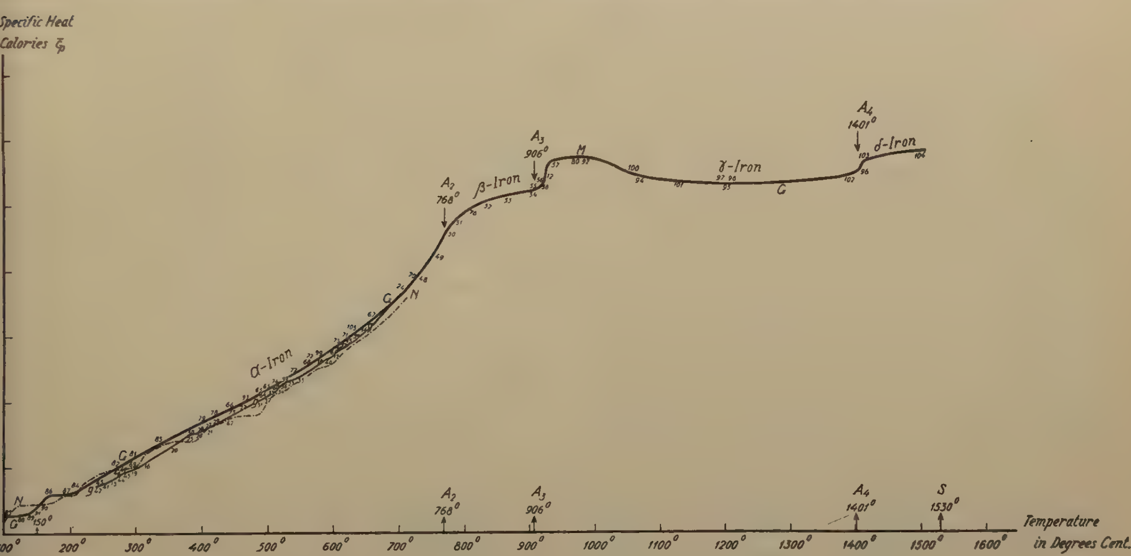

Fig. 1. *The Mean Specific Heats of Iron between 25° and 1500° C.*

1) W. ROBERTS–AUSTIN, Instit. Mech. Engin., 35 (1899); A. MÜLLER, Metallurgie, 6, 145 (1909).

TABLE I.

The Mean Specific Heats of Iron between 25° and 1500° C.

Sequence-Number of Experiment:	Temp. t in °C.:	Final temp. t' of the Calorimeter:	Quantity of Heat Q given off by 1 Gr. of iron between $t°$ and 25° in Calories:	Mean Specific Heat $\bar{c}_p$ between t en t' in Calories:
12	915°.9	25°.389	146.467	0.1644
13	253.6	22.418	26.880	0.1177
14	651.0	23.423	88.528	0.1414
15	615.9	24.128	81.846	0.1385
16	301.2	22.996	33.186	0.1202
17	598.8	22.739	78.684	0.1371
18	580.2	23.615	75.537	0.1361
19	285.2	22.622	31.003	0.1192
20	352.0	22.494	40.320	0.1233
21	402.4	22.673	47.416	0.1256
22	452.4	22.423	55.028	0.1287
23	519.0	23.092	65.460	0.1325
24	707.9	24.458	100.176	0.1467
25	380.2	23.323	44.454	0.1252
26	390.2	23.050	45.769	0.1253
27	395.3	22.903	46.581	0.1258
28	395.4	22.113	46.564	0.1257
29	402.4	22.155	47.446	0.1257
30	390.2	22.559	45.748	0.1253
31	480.2	22.978	59.144	0.1299
32	490.5	23.251	60.637	0.1302
33	500.4	23.453	62.353	0.1312
34	510.5	22.709	63.763	0.1313
35	548.8	23.316	70.060	0.1338
36	515.0	23.471	64.781	0.1322
37	610.0	24.104	80.673	0.1379
38	604.2	24.259	79.902	0.1380
39	628.2	24.504	86.908	0.1395
40	590.0	22.626	77.082	0.1364
40*a*	590.0	24.342	77.265	0.1367
41	238 4	23.400	25.041	0.1173
42	238.4	22.941	25.019	0.1172
43	238.4	23.021	25.089	0.1176
44	265.2	22.769	28.410	0.1183
45	276.3	21.666	29.917	0.1190
46	270.7	22.183	29.260	0.1191
47	276.3	22.314	29.961	0.1192

TABLE I. (*Continued*).
The Mean Specific Heats of Iron between 25° and 1500° C.

Sequence-Number of Experiment:	Temp. t in °C.:	Final temp. t' of the Calorimeter:	Quantity of Heat Q given off by 1 Gr. of iron between $t°$ and 25° in Calories:	Mean Specific Heat $\bar{c}_p$ between t en t' in Calories:
48	731.6°	24.187°	105.415	0.1492
49	749.7	25.159	110.205	0.1521
50	770.3	25.442	116.057	0.1557
51	780.2	24.904	118.476	0.1569
52	825.1	24.846	127.763	0.1597
53	850 4	24.887	132.624	0.1607
54	900.2	25.157	141.720	0.1619
55	905.3	25.089	142 745	0.1621
56	910.1	25.260	143.651	0.1623
57	930.7	25.451	150.430	0.1661
58	910.1	25.995	143.460	0.1621
59	505.6	23.962	63.273	0.1317
60	510.6	23.815	64.097	0.1320
61	604.3	24.089	80.059	0.1382
62	424.3	23.425	50.848	0.1273
63	492.9	23.541	60.081	0.1314
64	491.2	22.749	61.030	0.1309
65	491.2	23.141	61.273	0.1314
66	424.3	23.099	51.136	0.1281
67	670.1	24.115	92.344	0.1431
68	549.4	24.179	70.769	0.1350
69	282.8	23.029	30.894	0.1198
70	720.3	23.767	102.810	0.1479
71	610.8	24.235	81.484	0.1391
72	549.4	24.115	70.870	0.1351
73	425.3	22.553	51.007	0.1274
74	501.5	23.119	62.829	0.1319
75	605.0	23.172	80.397	0.1386
76	798.5	25.015	122.806	0.1588
77	530.6	22.664	67.506	0.1335
78	404.9	22.388	48.212	0.1269
79	383.5	22.472	45.196	0.1261
80	961.6	25.027	155.957	0.1665
81	287.4	23.472	31.808	0.1212
82	270.9	22.493	29.600	0.1204
83	331.0	22.295	37.693	0.1232
84	200.2	21.861	20.344	0.1161

TABLE I. *(Continued).*
The Mean Specific Heats of Iron between 25° and 1500° C.

Sequence-Number of Experiment:	Temp. t in °C.:	Final temp. t' of the Calorimeter:	Quantity of Heat Q given off by 1 Gr. of iron between $t°$ and 25° in Calories:	Mean Specific Heat $\bar{c}_p$ between t en t' in Calories:
85	100°.3	21°.267	8.489	0.1127
86	161.7	21.164	15.847	0.1159
87	191.8	21.232	19.299	0.1157
88	116.0	20.988	10.244	0.1126
89	130.2	21.825	11.861	0.1127
90	149.2	21.399	14.192	0.1143
91	137.9	22.092	12.812	0.1135
92	979.6	24.434	158.964	0.1665
93	471.1	23.025	58.219	0.1305
94	1067.5	24.144	171.340	0.1644
95	1197.2	24.421	191.352	0.1633
96	1403.9	25.032	229.227	0.1662
97	1197.1	24.267	191.730	0.1635
98	1197.2	24.212	191.659	0.1635
99	580.4	21.829	76.090	0.1370
100	1061.4	24.578	170.727	0.1647
101	1130.6	25.023	181.704	0.1643
102	1391.1	25.375	225.470	0.1651
103	1409.3	25.251	231.090	0.1669
104	1491.1	25.598	247.111	0.1686
105	630.4	22.935	84.998	0.1404

from the atmosphere and *hydrogen* set free in the electrolysis. This experience so far corroborates the explanation here suggested.

§ 4. The experimental data obtained, with the still gas-containing (12—58) as well as with the pure gas-free metal (59—105), are collected in *Table I*; the first **11** measurements were discarded, as only referring to not yet quite reproducible values of c_p.

The results are graphically represented in Fig. 1.

§ 5. From these data the values of the *true* specific heats c_p of *iron* between 100° and 1500° C. were graphically determined; the values obtained are collected in *Table II*.

Between 250° and 600° C. these values can, with a fair degree of accuracy, be represented by the formula:

$$c_p = 0{,}1335 + 1{,}5406.10^{-5}.(t-250) + 5{,}289.10^{-7}(t-250)^2 - 6{,}51.10^{-10}.(t-250)^3;$$

so that, in this range of temperatures, the *atomic heats* C_p are given by the equation:

$$C_p = 7{,}455 + 8{,}603.10^{-4} . (t-250) + 2{,}9534.10^{-5} . (t-250)^2 - 3{,}635.10^{-8} . (t-250)^3.$$

Some values of C_p for *α-iron* thus calculated are given in *Table III*. The c_p-t-curve of *iron* is graphically represented in Fig. 2.

TABLE II.
The true Specific Heats c_p of Iron between 100° and 1500° C.
(graphically determined).

Temperature: t in °C.	True Specific Heats c_p:	Temperature t in °C:	True Specific Heats c_p:
105°	0.112	555°	0.169
110	0.113	575	0.171
120	0.113	600	0.175
130	0.114	610	0.177
140	0.118	620	0.179
145	0.124	630	0.180
150	0.132	640	0.183
155	0.139	650	0.186
160	0.136	660	0.192
165	0.125	670	0.193
170	0.115	680	0.198
175	0.111	690	0.204
180	0.111	700	0.213
185	0.112	710	0.218
190	0.118	720	0.227
195	0.122	730	0.242
200	0.123	740	0.259
205	0.125	750	0.281
210	0.128	755	0.287
220	0.128	760	0.288
230	0.132	765	0.280
240	0.132	770	0.270
250	0.134	775	0.260
260	0.135	780	0.250
300	0.136	785	0.238
350	0.143	790	0.225
400	0.146	795	0.215
450	0.151	800	0.213
500	0.160	810	0.205
525	0.164	820	0.199

TABLE II. (*Continued*).
The true Specific Heats c_p of Iron between 100° and 1550° C.
(graphically determined).

Temperature t in °C.:	True Specific Heats c_p:	Temperature t in C°:	True Specific Heats c_p:
830	0.195	1150	0.156
840	0.192	1175	0.159
850	0.190	1200	0.162
860	0.188	1225	0.165
870	0.187	1250	0.169
880	0.185	1275	0.173
890	0.183	1300	0.176
900	0.182	1325	0.177
975	0.142	1350	0.178
1000	0.144	1375	0.179
1025	0.146	1400	0.180
1050	0.148	1425	0.199
1075	0.149	1450	0.200
1100	0.151	1475	0.200
1125	0.153	1500	0.200

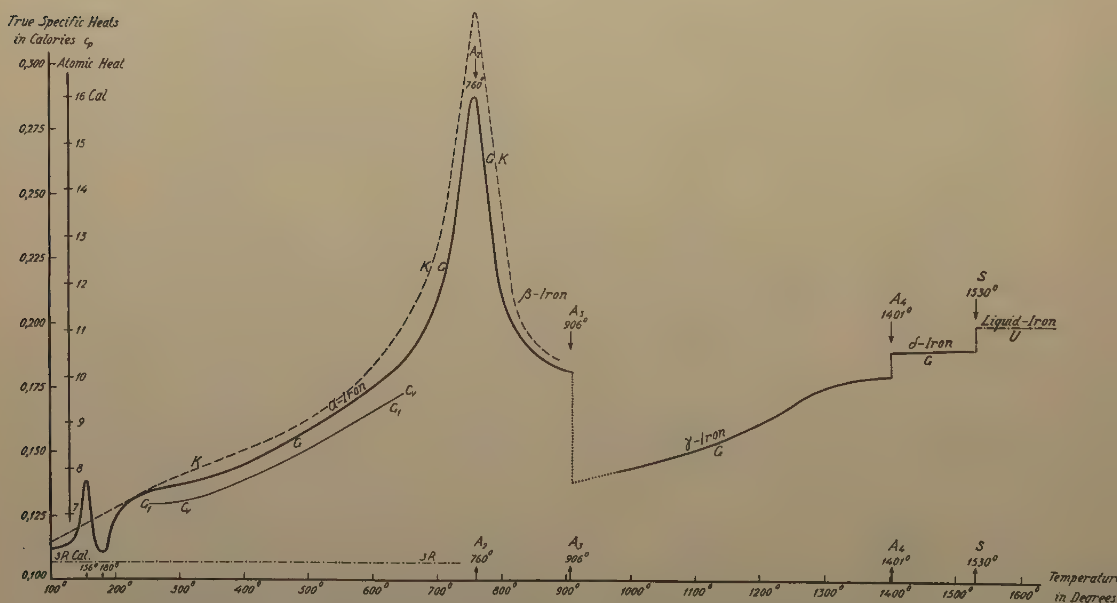

Fig. 2. *The True Specific Heats c_p of Iron between 25° and 1600° C.*

TABLE III:
The Atomic Heats C_p and (approximately) C_v of α-Iron at Different Temperatures between 250° and 650° C.

Temperature t in °C.:	Atomic Heats C_p:	Atomic Heats C_v: (approximately) *)
250°	7.455	7.209
300	7.572	7.276
350	7.807	7.471
400	8.125	7.767
450	8.516	8.091
500	8.946	8.482
550	9.415	8.896
600	9.817	9.236
650	10.308	9.677

C_p rises to about 16 Calories at the *Curie*-point. The heat of transformation af $\alpha \rightleftarrows \beta$-iron is of the order of about 4.8 Calories pro gramme [1]).

In Fig. 2 the true specific heats c_p of γ- and δ-*iron* are also graphically represented, as calculated from the Q-values by taking into account the values of the *transformation*-heats at A_3 and A_4; namely: 3,2 cal./gr. at A_3 and 2,2 cal./gr. at A_4 [2]). In the same way the *heat of fusion* of the metal at 1530° C. was supposed to be: 65,6[5] cal./gr. and thus the values of c_p for molten *iron* could equally be included. From the figure it becomes evident that c_p for δ-*iron* and for *liquid iron* is practically *independent* of the temperature.

Evidently the CURIE-point is, — in full accordance with the results of

*) The compressibility of *iron* $\varkappa$ is about $5{,}87 \cdot 10^{-7}$ at 30° C. and about $5{,}93 \cdot 10^{-7}$ at 75° C.; the value of 3α varies from $39 \cdot 10^{-6}$ between 100° and 200° C. to $52{,}2 \cdot 10^{-6}$ between 600° and 700°. (P. W. BRIDGMAN, Proc. Amer. Acad. Arts Sciences, **58**, 174 (1923). P. HIDNERT, Sc. Pap. Bur. Stand., **17**, 616 (1922).

[1]) H. VON STEINWEHR and A. SCHULZE, Phys. Zeits., **36**, 421 (1935); Zeits. f. Metallkunde, **27**, 132 (1935).

[2]) The values of these heat-effects, as given in the literature, are rather divergent: at A_3 they range from 3,42 cal. (H. ESSER and W. BURGHARDT, Archiv. f. Eisenhüttenwesen, (1934—'35), 8; H. KLINKHARDT, loco cit., p. 193) to 6,2 cal. (S. UMINO, Sc. Rep. Tohoku Univ., **16**, 104 (1927); H. VON STEINWEHR and A. SCHULZE, loco cit., p. 423). At A_4 they oscillate between 1,86 cal. (S. UMINO, Sc. Rep. Tohoku Univ., **18**, 104 (1929); F. WÜST, Forsch. Arb. Ing. Wesen, (1918), Heft 204, 21) and 2,53 cal. (P. OBERHOFFER and W. GROSSE, Stahl und Eisen, **47**, 580 (1927). The values for the heat of fusion are: 64,38 cal./gr. (P. OBERHOFFER and W. GROSSE, loco cit.) and 65,65 cal./gr. (S. UMINO, loco cit.). the latter value was here used.

KLINKHARDT [1]), who determined the true specific heats c_p immediately by means of a vacuum-calorimeter, in which the metal was heated by electronic bombardment, — situated at 760° C. Also at a higher temperature the results here obtained from the differential function $\frac{dQ}{dt}$ prove to be in excellent agreement with those of the author mentioned.

§ 6. The c_p-t-curve of *α-iron* shows a still permanent discontinuity between about 130° and 190° C. which apparently has another character than the discontinuities due to the presence of hydrogen in the metal. MORRIS [2]) also stated a discontinuity in the magnetic behaviour of *α-iron*, ROBIN [2]) an acoustic (elastic) anomaly at 150° C.; and NAESER and ROTH, as well as ourselves, found the said discontinuity even in a sample as well exempted of gas as possible.

A wire of electrolytic *iron* heated to 150° C. and subsequently quenched, yielded an *X*-ray-spectrogram which in *no* respects proved to differ from that of *α-iron* at room-temperature. Then within the evacuated camera such a wire was heated to about 150° C. and, whilst hot, photographed with *iron-α*-radiation. The spectrogram thus obtained again proved to the *quite the same* as that of *α-iron*: apparently no change whatsoever of the structure had occurred. As in some of the *nickel-iron*-alloys studied also indications of a discontinuity at 150° C. were found, the question about its nature must later-on be reconsidered more in detail.

Finally attenion here can be drawn to the fact that, — as may be seen from Fig. 2, — the values of the atomic heats C_p of solid *iron* between 100° up to the melting-point, and even those of the molten metal, far surpass the limit of $3R$ calories, as they reach values of more than eleven calories with all subsequent modifications of this element: thus DULONG and PETIT's "law" does not even approximately hold for *iron* at as low a temperature as 100° C.

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1) H. KLINKHARDT, Ann. d. Physik, (4), **84**, 167 (1927).

2) D. K. MORRIS, Phil. Mag., (5), **44**, 213 (1897); ROBIN, Compt. rend. Paris, **150**, 780 (1910).

Mathematics. — *Note über das Produkt $M_{k,m}(z)\, M_{-k,m}(z)$.* Von C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of February 26, 1938.)

In einer vorigen Arbeit [1]) habe ich unter gewissen Voraussetzungen für die Produkte $W_{k,m}(z)\, M_{-k,m}(z)$ und $W_{k,m}(z)\, W_{-k,m}(z)$ die folgenden Integraldarstellungen abgeleitet

$$W_{k,m}(z)\, M_{-k,m}(z) = \frac{z\, \Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)} \int_0^\infty J_{2m}(z \sinh t) \coth^{2k} \tfrac{1}{2} t\, dt,$$

$$W_{k,m}(z)\, W_{-k,m}(z) = \frac{2z}{\pi} \int_0^{\frac{1}{2}\pi} K_{2m}(z \sec\varphi) \cos 2k\varphi \sec\varphi\, d\varphi.$$

Diese Beziehungen sind Erweiterungen der bekannten Relationen

$$K_\nu(z)\, I_\nu(z) = \int_0^\infty J_{2\nu}(2z \sinh t)\, dt,$$

$$K_\nu^2(z) = 2 \int_0^{\frac{1}{2}\pi} K_{2\nu}(2z \sec\varphi) \sec\varphi\, d\varphi = 2 \int_0^\infty K_{2\nu}(2z \cosh t)\, dt.$$

Es liegt nahe zu versuchen für das NEUMANNsche Integral [2])

$$I_\nu^2(z) = \frac{1}{\pi} \int_{-\infty}^\infty I_{2\nu}(2z\, \mathrm{sech}\, t)\, \mathrm{sech}\, t\, dt = \frac{1}{\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} I_{2\nu}(2z \cos\varphi)\, d\varphi \quad . \quad (1)$$

eine analoge Erweiterung herzuleiten. Das Resultat lautet wie folgt:

Ist $\Re(\frac{1}{2}+m \pm k) > 0$, *so gilt*

$$M_{k,m}(z)\, M_{-k,m}(z) = \frac{z\, \Gamma^2(1+2m)}{\Gamma(\frac{1}{2}+m+k)\, \Gamma(\frac{1}{2}+m-k)} \int_{-\infty}^\infty I_{2m}(z\, \mathrm{sech}\, t)\, e^{2kt}\, \mathrm{sech}\, t\, dt \quad . \quad (2)$$

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1) C. S. MEIJER, Noch einige Integraldarstellungen für Produkte von WHITTAKERschen Funktionen, Proc. Royal Neth. Acad. Amsterdam, **40**, S. 871—879 (1937).

2) C. NEUMANN, Theorie der BESSEL'schen Functionen, S. 70 (1867); siehe auch G. N. WATSON, A treatise on the theory of BESSEL functions, S. 32, 150 und 441 (1922). In (1) ist $\Re(\nu) > -\frac{1}{2}$ und z beliebig.

Beweis. Ersetzt man in der bekannten Beziehung

$$\int_0^\infty \frac{u^{\lambda+k-1}\,du}{(u+1)^{2\lambda}} = \frac{\Gamma(\lambda+k)\,\Gamma(\lambda-k)}{\Gamma(2\lambda)} \qquad (\Re(\lambda\pm k)>0)$$

die Integrationsveränderliche u durch e^{2t}, so erhält man

$$2^{1-2\lambda}\int_{-\infty}^{\infty} \frac{e^{2kt}\,dt}{\cosh^{2\lambda} t} = \frac{\Gamma(\lambda+k)\,\Gamma(\lambda-k)}{\Gamma(2\lambda)}.$$

Die rechte Seite von (2) ist also gleich [3])

$$\frac{2^{-2m} z^{1+2m}\,\Gamma^2(1+2m)}{\Gamma(\frac{1}{2}+m+k)\,\Gamma(\frac{1}{2}+m-k)} \sum_{r=0}^{\infty} \frac{2^{-2r} z^{2r}}{\Gamma(1+r)\,\Gamma(1+2m+r)} \int_{-\infty}^{\infty} \frac{e^{2kt}\,dt}{(\cosh t)^{1+2m+2r}}$$

$$= \frac{z^{1+2m}\,\Gamma^2(1+2m)}{\Gamma(\frac{1}{2}+m+k)\,\Gamma(\frac{1}{2}+m-k)} \sum_{r=0}^{\infty} \frac{z^{2r}\,\Gamma(\frac{1}{2}+m+k+r)\,\Gamma(\frac{1}{2}+m-k+r)}{\Gamma(1+r)\,\Gamma(1+2m+r)\,\Gamma(1+2m+2r)}$$

$$= z^{1+2m}\,{}_2F_3(\tfrac{1}{2}+m+k,\ \tfrac{1}{2}+m-k;\ 1+2m,\ \tfrac{1}{2}+m,\ 1+m;\ \tfrac{1}{4}z^2).$$

Der letzte Ausdruck ist aber gleich [4]) $M_{k,m}(z)\,M_{-k,m}(z)$, so dass der Beweis von (2) geliefert ist.

Anwendungen. Man hat [5])

$$I_\nu^2(z) = \frac{1}{2^{1+4\nu} z\,\Gamma^2(1+\nu)}\,M_{0,\nu}^2(2z).$$

Formel (1) ist daher ein Spezialfall von (2).

Bezeichnet $D_n(z)$ die parabolische Zylinderfunktion, so gelten die folgenden Beziehungen [6])

$$\left.\begin{aligned} &\{D_n(ze^{\frac{1}{4}\pi i}) + D_n(ze^{-\frac{1}{4}\pi i})\}\{D_{-n-1}(ze^{\frac{1}{4}\pi i}) + D_{-n-1}(ze^{-\frac{1}{4}\pi i})\} \\ &\quad = \frac{4\pi z^{-1}}{\Gamma(\frac{1}{2}-\frac{1}{2}n)\,\Gamma(1+\frac{1}{2}n)}\,M_{\frac{1}{2}n+\frac{1}{4},-\frac{1}{4}}(\tfrac{1}{2}z^2)\,M_{-\frac{1}{2}n-\frac{1}{4},-\frac{1}{4}}(\tfrac{1}{2}z^2), \end{aligned}\right\} \quad (3)$$

$$\left.\begin{aligned} &\{D_n(ze^{\frac{1}{4}\pi i}) - D_n(ze^{-\frac{1}{4}\pi i})\}\{D_{-n-1}(ze^{\frac{1}{4}\pi i}) - D_{-n-1}(ze^{-\frac{1}{4}\pi i})\} \\ &\quad = -\frac{16\pi z^{-1}}{\Gamma(-\frac{1}{2}n)\,\Gamma(\frac{1}{2}+\frac{1}{2}n)}\,M_{\frac{1}{2}n+\frac{1}{4},\frac{1}{4}}(\tfrac{1}{2}z^2)\,M_{-\frac{1}{2}n-\frac{1}{4},\frac{1}{4}}(\tfrac{1}{2}z^2). \end{aligned}\right\} \quad (4)$$

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[3]) Gliedweise Integration ist gestattet; man vergl. T. J. I'A. BROMWICH, An introduction to the theory of infinite series (second edition, 1926), § 176 B.

[4]) Man vergl. C. S. MEIJER, Ueber Produkte von WHITTAKERschen Funktionen, Proc. Royal Neth. Acad. Amsterdam, **40**, S. 133—141 und S. 259—263, Formeln (9) und (2) (1937).

[5]) E. T. WHITTAKER and G. N. WATSON, A course of modern analysis (fourth edition, 1927), § 17. 212.

[6]) Man vergl. MEIJER, loc. cit. [4]), Formeln (32), (33) und (2).

Nun hat man [7])

$$I_{-\frac{1}{2}}(w) = \sqrt{\frac{2}{\pi w}}\, \cosh w, \quad I_{\frac{1}{2}}(w) = \sqrt{\frac{2}{\pi w}}\, \sinh w.$$

Aus (3) und (2) ergibt sich somit, falls $-1 < \Re(n) < 0$ ist,

$$\{D_n(z e^{\frac{1}{4}\pi i}) + D_n(z e^{-\frac{1}{4}\pi i})\}\{D_{-n-1}(z e^{\frac{1}{4}\pi i}) + D_{-n-1}(z e^{-\frac{1}{4}\pi i})\}$$

$$= -\frac{2 \sin n\pi}{\sqrt{\pi}} \int_{-\infty}^{\infty} \cosh(\tfrac{1}{2} z^2 \operatorname{sech} t)\, e^{(n+\frac{1}{2})t} \operatorname{sech}^{\frac{1}{2}} t\, dt;$$

ebenso aus (4) und (2), falls $-2 < \Re(n) < 1$ ist,

$$\{D_n(z e^{\frac{1}{4}\pi i}) - D_n(z e^{-\frac{1}{4}\pi i})\}\{D_{-n-1}(z e^{\frac{1}{4}\pi i}) - D_{-n-1}(z e^{-\frac{1}{4}\pi i})\}$$

$$= \frac{2 \sin n\pi}{\sqrt{\pi}} \int_{-\infty}^{\infty} \sinh(\tfrac{1}{2} z^2 \operatorname{sech} t)\, e^{(n+\frac{1}{2})t} \operatorname{sech}^{\frac{1}{2}} t\, dt.$$

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7) WATSON, loc. cit., S. 80, Formeln (10) und (11).

Mathematics. — *Ueber Differentialkovarianten erster Ordnung der binären kubischen und der binären biquadratischen Differentialform.* Von P. G. MOLENAAR. (Communicated by Prof. R. WEITZENBÖCK.)

(Communicated at the meeting of February 26, 1938.)

Die Elimination der Γ-Symbolen aus den Gleichungen für die kovariante erste Ableitung einer Differentialform zur Aufsuchung von Differentialkovarianten ist ein verwickeltes Problem [1]).

Es wird hier eine Methode angegeben zur Durchführung dieser Elimination in zwei speziellen Fällen.

§ 1 gibt das Eliminationsresultat für die binäre kubische Differentialform $f = a_{ikl}\, dx^i\, dx^k\, dx^l$.

In § 2 wird aus diesem Resultat mit Hilfe des Satzes von GRAM eine Differentialkovariante erster Ordnung und zweiter Stufe abgeleitet.

§ 3 bringt diese Differentialkovariante auf eine bequemere Form.

§ 4 stellt dieselbe Frage für die binäre biquadratische Differentialform $f = a_{iklm}\, dx^i dx^k dx^l dx^m$.

In § 5 wird die Elimination ersetzt durch die Aufstellung einer Determinante, worauf im § 6 dann eine Differentialkovariante erster Ordnung und siebenter Stufe für die biquadratische Differentialform bestimmt wird.

§ 1. Die binäre kubische Differentialform

$$f = a_{ikl}\, dx^i dx^k dx^l = a^3_{dx}$$

hat drei Differentialkomitanten nullter Ordnung:

$$\triangle = \alpha^2_{dx} = (ab)^2 a_{dx} b_{dx} \quad Q = (ab)^2 (ca)\, b_{dx} c^2_{dx} \quad R = (\alpha\beta)^2 = (ab)^2 (cd)^2 (ad)(bc).$$ [2])

Die kovariante erste Ableitung wird definiert durch folgende acht Gleichungen:

$$a_{ikl,m} = \frac{\partial a_{ikl}}{\partial x_m} - a_{\lambda kl}\, \Gamma^{\lambda}_{im} - a_{i\lambda l}\, \Gamma^{\lambda}_{km} - a_{ik\lambda}\, \Gamma^{\lambda}_{lm}.$$

Die sechs Γ-Symbole lassen sich folgendermassen eliminieren.

Sei

$$a_{\lambda kl}\, \Gamma^{\lambda}_{im} + a_{i\lambda l}\, \Gamma^{\lambda}_{km} + a_{ik\lambda}\, \Gamma^{\lambda}_{lm} = d_{ikl,m} \quad \ldots\ldots \quad (1)$$

also

$$a_{ikl,m} = \frac{\partial a_{ikl}}{\partial x_m} - d_{ikl,m} \quad \ldots\ldots\ldots \quad (2)$$

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[1]) Vgl. R. WEITZENBÖCK, Proc. Royal Neth. Acad. Amsterdam, **29**, 400—403 (1926).

[2]) Vgl. GORDAN–KERSCHENSTEINER, Invariantentheorie II, S. 167.

dann ist es notwendig und hinreichend, acht Koeffizienten $\eta_1\, \eta_2 \ldots \eta_8$ derart zu bestimmen, dass

$$\eta_1\, d_{111,1} + \eta_2\, d_{111,2} + \ldots + \eta_7\, d_{222,1} + \eta_8\, d_{222,2} \equiv 0 \quad \{\Gamma^l_{ik}\}. \qquad (3)$$

wird, dass also nach (1)

$$\begin{aligned}
&\eta_1 \,.\, 3\, a_{\lambda 11}\, \Gamma^\lambda_{11} && + \eta_2 \,.\, 3\, a_{\lambda 11}\, \Gamma^\lambda_{12} && + \\
&+ \eta_3 (2\, a_{\lambda 12}\, \Gamma^\lambda_{11} + a_{\lambda 11}\, \Gamma^\lambda_{12}) && + \eta_4 (2\, a_{\lambda 12}\, \Gamma^\lambda_{12} + a_{\lambda 11}\, \Gamma^\lambda_{22}) && + \\
&+ \eta_5 (a_{\lambda 22}\, \Gamma^\lambda_{11} + 2\, a_{\lambda 12}\, \Gamma^\lambda_{12}) && + \eta_6 (a_{\lambda 22}\, \Gamma^\lambda_{12} + 2\, a_{\lambda 12}\, \Gamma^\lambda_{22}) && + \\
&+ \eta_7 \,.\, \quad 3\, a_{\lambda 22}\, \Gamma^\lambda_{12} && + \eta_8 \,.\, \quad 3\, a_{\lambda 22}\, \Gamma^\lambda_{22} && \equiv \{\Gamma^l_{ik}\}.
\end{aligned}$$

Die acht η's müssen also folgenden acht Gleichungen genügen:

$$\begin{aligned}
&3\, a_{111}\, \eta_1 + 2\, a_{112}\, \eta_3 + a_{122}\, \eta_5 = 0 \\
&3\, a_{112}\, \eta_1 + 2\, a_{122}\, \eta_3 + a_{222}\, \eta_5 = 0 \\
&3\, a_{111}\, \eta_2 + a_{111}\, \eta_3 + 2\, a_{112}\, \eta_4 + 2\, a_{112}\, \eta_5 + a_{122}\, \eta_6 + 3\, a_{122}\, \eta_7 = 0 \\
&3\, a_{112}\, \eta_2 + a_{112}\, \eta_3 + 2\, a_{122}\, \eta_4 + 2\, a_{122}\, \eta_5 + a_{222}\, \eta_6 + 3\, a_{222}\, \eta_7 = 0 \\
&a_{111}\, \eta_4 + 2\, a_{112}\, \eta_6 + 3\, a_{122}\, \eta_8 = 0 \\
&a_{112}\, \eta_4 + 2\, a_{122}\, \eta_6 + 3\, a_{222}\, \eta_8 = 0
\end{aligned}$$

oder, anders geschrieben:

$$\begin{aligned}
&3\, a_{111}\, \eta_1 + 2\, a_{112}\, \eta_3 = - a_{122}\, \eta_5 \\
&3\, a_{112}\, \eta_1 + 2\, a_{122}\, \eta_3 = - a_{222}\, \eta_5 \\
&3\, a_{111}\, \eta_2 + a_{111}\, \eta_3 + a_{122}\, \eta_6 + 3\, a_{122}\, \eta_7 = - 2\, a_{112}\, \eta_4 - 2\, a_{112}\, \eta_5 \\
&3\, a_{112}\, \eta_2 + a_{112}\, \eta_3 + a_{222}\, \eta_6 + 3\, a_{222}\, \eta_7 = - 2\, a_{122}\, \eta_4 - 2\, a_{122}\, \eta_5 \\
&2\, a_{112}\, \eta_6 + 3\, a_{122}\, \eta_8 = - a_{111}\, \eta_4 \\
&2\, a_{122}\, \eta_6 + 3\, a_{222}\, \eta_8 = - a_{112}\, \eta_4
\end{aligned}$$

Man findet dann:

$$\eta_i = \frac{\Delta_i}{\Delta} \; (i = 1, 2, 3, 6, 7, 8)$$

worin

$$\Delta = \begin{vmatrix}
3\, a_{111} & 0 & 2\, a_{112} & 0 & 0 & 0 \\
3\, a_{112} & 0 & 2\, a_{122} & 0 & 0 & 0 \\
0 & 3\, a_{111} & a_{111} & a_{122} & 3\, a_{122} & 0 \\
0 & 3\, a_{112} & a_{112} & a_{222} & 3\, a_{222} & 0 \\
0 & 0 & 0 & 2\, a_{112} & 0 & 3\, a_{122} \\
0 & 0 & 0 & 2\, a_{122} & 0 & 3\, a_{222}
\end{vmatrix}$$

oder

$$\triangle = 324\,(a_{111}\,a_{122} - a_{112}^2)\,(a_{112}\,a_{222} - a_{122}^2)\,(a_{111}\,a_{222} - a_{112}\,a_{122})$$

Aber

$$\alpha_{11} = 2\,(a_{111}\,a_{122} - a_{112}^2)$$
$$\alpha_{12} = \quad a_{111}\,a_{222} - a_{112}\,a_{122}$$
$$\alpha_{22} = 2\,(a_{112}\,a_{222} - a_{122}^2)$$

also

$$\triangle = 81\,\alpha_{11}\,\alpha_{12}\,\alpha_{22}.$$

Ferner ist

$$\triangle_1 = \begin{vmatrix} -\quad a_{122}\,\eta_5 & 0 & 2\,a_{112} & 0 & 0 & 0 \\ -\quad a_{222}\,\eta_5 & 0 & 2\,a_{122} & 0 & 0 & 0 \\ -2\,a_{112}\,\eta_4 - 2\,a_{112}\,\eta_5 & 3\,a_{111} & a_{111} & a_{122} & 3\,a_{122} & 0 \\ -2\,a_{122}\,\eta_4 - 2\,a_{122}\,\eta_5 & 3\,a_{112} & a_{112} & a_{222} & 3\,a_{222} & 0 \\ -\quad a_{111}\,\eta_4 & 0 & 0 & 2\,a_{112} & 0 & 3\,a_{122} \\ -\quad a_{112}\,\eta_4 & 0 & 0 & 2\,a_{122} & 0 & 3\,a_{222} \end{vmatrix}$$

oder

$$\triangle_1 = -\,108\,\eta_5\,(a_{122}^2 - a_{112}\,a_{222})\,(a_{111}\,a_{222} - a_{112}\,a_{122})\,(a_{112}\,a_{222} - a_{122}^2)$$

also

$$\triangle_1 = \qquad\qquad + 27\,\alpha_{12}\,\alpha_{22}^2\,\eta_5.$$

Ebenso findet man

$$\triangle_2 = -\,27\,\alpha_{11}\,\alpha_{22}^2\,\eta_4 \qquad + 27\,\alpha_{22}\,(\alpha_{12}^2 - \alpha_{11}\,\alpha_{22})\,\eta_5$$
$$\triangle_3 = \qquad\qquad - 81\,\alpha_{12}^2\,\alpha_{22}\,\eta_5$$
$$\triangle_6 = -\,81\,\alpha_{12}^2\,\alpha_{11}\,\eta_4$$
$$\triangle_7 = +\,27\,\alpha_{11}\,(\alpha_{12}^2 - \alpha_{11}\,\alpha_{22})\,\eta_4 - 27\,\alpha_{11}^2\,\alpha_{22}\,\eta_5$$
$$\triangle_8 = +\,27\,\alpha_{11}^2\,\alpha_{12}\,\eta_4.$$

Setzt man erst $\eta_4 = 3\,\alpha_{12}\alpha_{22}$ und $\eta_5 = 0$ und dann $\eta_4 = 0$ und $\eta_5 = 3\,\alpha_{11}\,\alpha_{12}$, so erhält man zwei linear unabhängige Lösungen I und II:

	η_1	η_2	η_3	η_4	η_5	η_6	η_7	η_8
I	0	$-\alpha_{22}^2$	0	$3\,\alpha_{12}\,\alpha_{22}$	0	$-3\,\alpha_{12}^2$	$\alpha_{12}^2 - \alpha_{11}\,\alpha_{22}$	$\alpha_{11}\,\alpha_{12}$
II	$\alpha_{12}\,\alpha_{22}$	$\alpha_{12}^2 - \alpha_{11}\,\alpha_{22}$	$-3\,\alpha_{12}^2$	0	$3\,\alpha_{11}\,\alpha_{12}$	0	$-\alpha_{11}^2$	0

Die lineare Kombination $\dfrac{\alpha_{11}\,I - \alpha_{22}\,II}{\alpha_{12}}$ gibt noch:

III	$-\alpha_{22}^2$	$-\alpha_{12}\,\alpha_{22}$	$3\alpha_{12}\,\alpha_{22}$	$3\alpha_{11}\,\alpha_{22}$	$-3\alpha_{11}\alpha_{22}$	$-3\alpha_{11}\,\alpha_{12}$	$\alpha_{11}\,\alpha_{12}$	α_{11}^2

§ 2. Wählt man für $\eta_1\, \eta_2 \ldots . \eta_8$ eine dieser Lösungen, so enthält die Form

$$\eta_1\, a_{111,1} + \eta_2\, a_{111,2} + \ldots . + \eta_8\, a_{222,2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

nach (2) und (3) keine Γ-Symbole, und ist identisch gleich an

$$\eta_1 \frac{\partial a_{111}}{\partial x_1} + \eta_2 \frac{\partial a_{111}}{\partial x_2} + \ldots . + \eta_8 \frac{\partial a_{222}}{\partial x_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substitution von I, II und III in (4) gibt dann die invarianten Formen:

$$A_1 = -\alpha_{22}^2 a_{111,2} + 3\alpha_{12}\, \alpha_{22}\, a_{112,2} - 3\alpha_{12}^2\, a_{122,2} + (\alpha_{12}^2 - \alpha_{11}\, \alpha_{22})\, a_{222,1} + \alpha_{11}\, \alpha_{12}\, a_{222,2}$$

$$A_2 = \alpha_{12}\, \alpha_{22}\, a_{111,1} + (\alpha_{12}^2 - \alpha_{11}\, \alpha_{22})\, a_{111,2} - 3\alpha_{12}^2\, a_{112,1} + 3\alpha_{11}\, \alpha_{12}\, a_{122,1} - \alpha_{11}^2\, a_{222,1}$$

$$A_3 = -\alpha_{22}^2\, a_{111,1} - \alpha_{12}\, \alpha_{22}\, a_{111,2} + 3\alpha_{12}\, \alpha_{22}\, a_{112,1} + 3\alpha_{11}\, \alpha_{22}\, a_{112,2} -$$
$$- 3\alpha_{11}\, \alpha_{22}\, a_{122,1} - 3\alpha_{11}\, \alpha_{12}\, a_{122,2} + \alpha_{11}\, \alpha_{12}\, a_{222,1} + \alpha_{11}^2\, a_{222,2}.$$

Jetzt kann man aus diesen invarianten Formen nach einem Satz von GRAM eine Kovariante bilden [3]). Hierzu stellen wir die kovariante erste Ableitung von f symbolisch dar durch:

$$a_{ikl,m} = \varphi_{ikl}\, \psi_m = \varphi_i\, \varphi_k\, \varphi_l\, \psi_m.$$

Sind ferner $\alpha_{ik} = \alpha_i\, \alpha_k$ und $\beta_{ik} = \beta_i\, \beta_k$ aequivalente Symbole, so wird

$$A_1 = -\alpha_2^2 \beta_2^2 \varphi_1^3 \psi_2 + 3\alpha_1\, \alpha_2\, \beta_2^2 \varphi_1^2 \varphi_2 \psi_2 - 3\alpha_1\, \alpha_2\, \beta_1\, \beta_2\, \varphi_1\, \varphi_2^2 \psi_2 +$$
$$+ (\alpha_1\, \alpha_2\, \beta_1\, \beta_2 - \alpha_1^2 \beta_2^2)\, \varphi_2^3 \psi_1 + \alpha_1^2 \beta_1\, \beta_2\, \varphi_2^3 \psi_2.$$

Aus diesem Leitglied entsteht dann durch die lineare Transformation

$$\bar{a}_1 = a_{dx} \quad \bar{a}_2 = a_{dy} \quad \text{u. s. w.}$$

die Kovariante (wenn man der Einfachheit halber dx und dy durch x und y ersetzt und Potenzen von (xy) wegläszt):

$$K = -\alpha_y^2\, \beta_y^2\, \varphi_x^3\, \psi_y + 3\, \alpha_x\, \alpha_y\, \beta_y^2\, \varphi_x^2\, \varphi_y\, \psi_y - 3\alpha_x\, \alpha_y\, \beta_x\, \beta_y\, \varphi_x\, \varphi_y^2\, \psi_y +$$
$$+ (\alpha_x\, \alpha_y\, \beta_x\, \beta_y - \alpha_x^2\, \beta_y^2)\, \varphi_y^3\, \psi_x + \alpha_x^2\, \beta_x\, \beta_y\, \varphi_y^3\, \psi_y.$$

Durch Umformen gelingt es hier, einen Faktor $(xy)^3$ abzuspalten; man erhält nach einiger Rechnung:

$$K = (xy)^3 \left\{ \tfrac{1}{2} (\alpha\beta)^2\, \varphi_y^2\, (\varphi\psi) - (\varphi\beta)^2\, (\varphi\alpha)\, \psi_y\, \alpha_y \right\}$$

Somit ist

$$M = m_{dx}^2 = \tfrac{1}{2}\, R\, (\varphi\psi)\, \varphi_{dx}^2 - (\varphi\beta)^2\, (\varphi\alpha)\, \psi_{dx}\, \alpha_{dx} = \tfrac{1}{2}\, RU - V \quad . \quad . \quad (6)$$

eine Differentialkovariante von $f = a_{dx}^3$.

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[3]) Vgl. GRAM, Mathem. Ann. 7 (1873) und R. WEITZENBÖCK, Invariantentheorie, S. 159.

Ihre Komponenten sind:

$$m_{ik} = \tfrac{1}{2} R (\varphi \psi) \varphi_i \varphi_k - \tfrac{1}{2} (\varphi \beta)^2 (\varphi a) (\psi_i a_k + \psi_k a_i) = \tfrac{1}{2} R u_{ik} - v_{ik}.$$

Man findet leicht:

$$m_{11} = -A_2 \qquad m_{22} = +A_1 \qquad m_{12} = \tfrac{1}{2} A_3.$$

M enthält daher keine Γ-Symbole.

Aus $A_3 = \dfrac{\alpha_{11} A_1 - \alpha_{22} A_2}{\alpha_{12}}$ folgt noch:

$$2 m_{12} \alpha_{12} = \alpha_{11} m_{22} + \alpha_{22} m_{11} \quad \text{oder} \quad (\alpha m)^2 = 0,$$

was auch unmittelbar aus (6) gefolgert werden kann.

m^2_{dx} ist bei $R \neq 0$ durch Einführung kontravarianter Komponenten auf eine bequemere Form zu bringen. Setzt man:

$$\alpha^{pq} = \frac{1}{|\alpha|} \cdot \frac{\partial |\alpha|}{\partial \alpha_{pq}} = \frac{2}{R} \alpha_{\underline{pq}} \quad \text{mit} \quad |\alpha| = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}$$

so wird

$$v_{ik} = \tfrac{1}{2} (\varphi \beta)^2 (\varphi a) (\psi_i a_k + \psi_k a_i) = \tfrac{1}{4} R \varphi_{pq} \alpha^{pq} (\varphi \beta) (\psi_i \beta_k + \psi_k \beta_i)$$

also

$$m_{ik} = \tfrac{1}{2} R \{ u_{ik} - \tfrac{1}{2} \varphi_{pq} \alpha^{pq} (\varphi \beta) (\psi_i \beta_k + \psi_k \beta_i) \} = \tfrac{1}{2} R \,.\, n_{ik}.$$

Es handelt sich dann um die Differentialkovariante N mit Komponenten:

$$n_{ik} = u_{ik} - w_{ik} = u_{ik} - \tfrac{1}{2} \varphi_{pq} \alpha^{pq} (\varphi \beta) (\psi_i \beta_k + \psi_k \beta_i).$$

§ 3. Da (4) und (5) identisch gleich sind, kann man $a_{ikl,m}$ durch $\dfrac{\partial a_{ikl}}{\partial x_m}$ ersetzen. Man findet:

$$u_{ik} = (\varphi \psi) \varphi_{ik} = \varphi_{ik1} \psi_2 - \varphi_{ik2} \psi_1 = a_{ik1,2} - a_{ik2,1} = \frac{\partial a_{ik1}}{\partial x_2} - \frac{\partial a_{ik2}}{\partial x_1}.$$

$$w_{ik} = \tfrac{1}{2} \alpha^{pq} \,.\, \left\{ \begin{vmatrix} \varphi_{pq1} \psi_i & \varphi_{pq2} \psi_i \\ \beta_{1k} & \beta_{2k} \end{vmatrix} + \begin{vmatrix} \varphi_{pq1} \psi_k & \varphi_{pq2} \psi_k \\ \beta_{1i} & \beta_{2i} \end{vmatrix} \right\} =$$

$$= \tfrac{1}{2} \alpha^{pq} \,.\, \left\{ \begin{vmatrix} \dfrac{\partial a_{pq1}}{\partial x_i} & \dfrac{\partial a_{pq2}}{\partial x_i} \\ \beta_{1k} & \beta_{2k} \end{vmatrix} + \begin{vmatrix} \dfrac{\partial a_{pq1}}{\partial x_k} & \dfrac{\partial a_{pq2}}{\partial x_k} \\ \beta_{1i} & \beta_{2i} \end{vmatrix} \right\} = \tfrac{1}{2} \alpha^{pq} \,.\, (H_{pq,ik} + H_{pq,ki}).$$

Es ist aber

$$\alpha^{pq} \,.\, H_{pq,ik} = \alpha^{pq} \,.\, \begin{vmatrix} \dfrac{\partial a_{pq1}}{\partial x_i} & \dfrac{\partial a_{pq2}}{\partial x_i} \\ \beta_{1k} & \beta_{2k} \end{vmatrix} =$$

$$= \begin{vmatrix} \dfrac{\partial (\alpha^{pq} a_{pq1})}{\partial x_i} & \dfrac{\partial (\alpha^{pq} a_{pq2})}{\partial x_i} \\ \beta_{1k} & \beta_{2k} \end{vmatrix} - \frac{\partial \alpha^{pq}}{\partial x_i} \,.\, \begin{vmatrix} a_{pq1} & a_{pq2} \\ \beta_{1k} & \beta_{2k} \end{vmatrix}.$$

Nun ist

$$(\alpha a)^2 a_x = 0 \text{ also } \alpha^{pq} a_{pq1} = \alpha^{pq} a_{pq2} = 0$$

daher wird

$$\alpha^{pq} \cdot H_{pq,ik} = -\frac{\partial \alpha^{pq}}{\partial x_i} \cdot \begin{vmatrix} a_{pq1} & a_{pq2} \\ \alpha_{1k} & \alpha_{2k} \end{vmatrix}.$$

Ferner ist $\alpha_{ik} = (ab)^2 a_i b_k$, also

$$\begin{vmatrix} a_{pq1} & a_{pq2} \\ \alpha_{1k} & \alpha_{2k} \end{vmatrix} = \begin{vmatrix} c_{pq1} & c_{pq2} \\ \alpha_{1k} & \alpha_{2k} \end{vmatrix} = c_{pq} (ab)^2 b_k \begin{vmatrix} c_1 & c_2 \\ a_1 & a_2 \end{vmatrix} = (ab)^2 (ca) b_k c_{pq} = Q_{k,pq}.$$

Die binäre kubische Differentialform hat also eine Differentialkovariante erster Ordnung und zweiter Stufe mit Komponenten:

$$n_{ik} = \left(\frac{\partial a_{ik1}}{\partial x_2} - \frac{\partial a_{ik2}}{\partial x_1}\right) + \tfrac{1}{2}\left(\frac{\partial \alpha^{pq}}{\partial x_i} Q_{k,pq} + \frac{\partial \alpha^{pq}}{\partial x_k} Q_{i,pq}\right). \qquad (R \neq 0)$$

§ 4. Die binäre *biquadratische* Differentialform

$$f = a_{iklm}\, dx^i\, dx^k\, dx^l\, dx^m = a_{dx}^4$$

hat vier Differentialkomitanten nullter Ordnung [4]):

$$\triangle = (ab)^2 a_{dx}^2 b_{dx}^2 \qquad t = (ab)^2 (cb) a_{dx}^2 b_{dx} c_{dx}^3$$

$$i = (ab)^4 \qquad j = (ab)^2 (bc)^2 (ca)^2.$$

Die kovariante erste Ableitung wird definiert durch folgende zehn Gleichungen:

$$a_{iklm,p} = \frac{\partial a_{iklm}}{\partial x_p} - a_{\lambda klm} \Gamma^\lambda_{ip} - a_{i\lambda lm} \Gamma^\lambda_{kp} - a_{ik\lambda m} \Gamma^\lambda_{lp} - a_{ikl\lambda} \Gamma^\lambda_{mp}.$$

Setzt man

$$a_{\lambda klm} \Gamma^\lambda_{ip} + a_{i\lambda lm} \Gamma^\lambda_{kp} + a_{ik\lambda m} \Gamma^\lambda_{lp} + a_{ikl\lambda} \Gamma^\lambda_{mp} = d_{iklm,p} \quad \ldots \quad (7)$$

so kann man zehn Koeffizienten $\eta_1\, \eta_2 \ldots \eta_{10}$ derart bestimmen, dass

$$\eta_1 d_{1111,1} + \eta_2 d_{1111,2} + \ldots \ldots \ldots + \eta_{10} d_{2222,2} = 0 \{\Gamma^l_{ik}\}$$

oder nach (7)

$$\begin{aligned}
&\eta_1 \cdot 4 a_{\lambda 111} \Gamma^\lambda_{11} && + \eta_2 \cdot 4 a_{\lambda 111} \Gamma^\lambda_{12} + \\
&+ \eta_3 (3 a_{\lambda 112} \Gamma^\lambda_{11} + a_{\lambda 111} \Gamma^\lambda_{12}) && + \eta_4 (3 a_{\lambda 112} \Gamma^\lambda_{12} + a_{\lambda 111} \Gamma^\lambda_{22}) + \\
&+ \eta_5 (2 a_{\lambda 122} \Gamma^\lambda_{11} + 2 a_{\lambda 112} \Gamma^\lambda_{12}) && + \eta_6 (2 a_{\lambda 122} \Gamma^\lambda_{12} + 2 a_{\lambda 112} \Gamma^\lambda_{22}) + \\
&+ \eta_7 (a_{\lambda 222} \Gamma^\lambda_{11} + 3 a_{\lambda 122} \Gamma^\lambda_{12}) && + \eta_8 (a_{\lambda 222} \Gamma^\lambda_{12} + 3 a_{\lambda 122} \Gamma^\lambda_{22}) + \\
&+ \eta_9 \cdot \quad 4 a_{\lambda 222} \Gamma^\lambda_{12} && + \eta_{10} \cdot \quad 4 a_{\lambda 222} \Gamma^\lambda_{22} = 0 \{\Gamma^l_{ik}\}.
\end{aligned}$$

4) Vgl. GORDAN–KERSCHENSTEINER, Invariantentheorie II, S. 178.

Die zehn η's müssen also folgenden sechs Gleichungen genügen:

$$\begin{aligned}
4\,a_{\lambda 111}\,\eta_1 + 3\,a_{\lambda 112}\,\eta_3 + 2\,a_{\lambda 122}\,\eta_5 + a_{\lambda 222}\,\eta_7 &= 0 \\
4\,a_{\lambda 111}\,\eta_2 + 3\,a_{\lambda 112}\,\eta_4 + 2\,a_{\lambda 122}\,\eta_6 + a_{\lambda 222}\,\eta_8 &+ \\
+\, a_{\lambda 111}\,\eta_3 + 2\,a_{\lambda 112}\,\eta_5 + 3\,a_{\lambda 122}\,\eta_7 + 4\,a_{\lambda 222}\,\eta_9 &= 0 \qquad (\lambda = 1, 2) \\
a_{\lambda 111}\,\eta_4 + 2\,a_{\lambda 112}\,\eta_6 + 3\,a_{\lambda 122}\,\eta_8 + 4\,a_{\lambda 222}\,\eta_{10} &= 0.
\end{aligned}$$

Hieraus kann man die η's berechnen. Die Form

$$\eta_1\, a_{1111,1} + \eta_2\, a_{1111,2} + \dots + \eta_{10}\, a_{2222,2}$$

enthält dann keine Γ-Symbole mehr. Durch eine lineare Transformation kann man dann wieder nach dem Satz von GRAM eine Kovariante konstruieren.

§ 5. Leichter aber kommt man hier auf folgende Weise zum Ziel. Die kovariante erste Ableitung wird symbolisch dargestellt durch:

$$a_{iklm,\,p} = \varphi_{iklm}\,\psi_p = \varphi_i\,\varphi_k\,\varphi_l\,\varphi_m\,\psi_p.$$

Jede Determinante der Matrix

$$m = \left\| \begin{matrix}
4a_{1111} & 0 & 3a_{1112} & 0 & 2a_{1122} & 0 & a_{1222} & 0 & 0 & 0 \\
4a_{1112} & 0 & 3a_{1122} & 0 & 2a_{1222} & 0 & a_{2222} & 0 & 0 & 0 \\
0 & 4a_{1111} & a_{1111} & 3a_{1112} & 2a_{1112} & 2a_{1122} & 3a_{1122} & a_{1222} & 4a_{1222} & 0 \\
0 & 4a_{1112} & a_{1112} & 3a_{1122} & 2a_{1122} & 2a_{1222} & 3a_{1222} & a_{2222} & 4a_{2222} & 0 \\
0 & 0 & 0 & a_{1111} & 0 & 2a_{1112} & 0 & 3a_{1122} & 0 & 4a_{1222} \\
0 & 0 & 0 & a_{1112} & 0 & 2a_{1122} & 0 & 3a_{1222} & 0 & 4a_{2222} \\
\varphi_{1111}\psi_1 & \varphi_{1111}\psi_2 & \varphi_{1112}\psi_1 & \varphi_{1112}\psi_2 & \varphi_{1122}\psi_1 & \varphi_{1122}\psi_2 & \varphi_{1222}\psi_1 & \varphi_{1222}\psi_2 & \varphi_{2222}\psi_1 & \varphi_{2222}\psi_2
\end{matrix} \right\|$$

ist dann frei von Γ-Symbolen. Wir wollen jetzt die Determinante K der letzten sieben Kolonnen der Transformation

$$\bar{a}_1 = a_{dx} \quad \bar{a}_2 = a_{dy}$$

unterwerfen. Man findet, wenn a, b, c, d, e und f aequivalente Symbole sind (und dx und dy durch x und y ersetzt werden):

$$K = \left| \begin{matrix}
0 & 2a_x^2 a_y^2 & 0 & a_x a_y^3 & 0 & 0 & 0 \\
0 & 2b_x b_y^3 & 0 & b_y^4 & 0 & 0 & 0 \\
3c_x^3 c_y & 2c_x^3 c_y & 2c_x^2 c_y^2 & 3c_x^2 c_y^2 & c_x c_y^3 & 4c_x c_y^3 & 0 \\
3d_x^2 d_y^2 & 2d_x^2 d_y^2 & 2d_x d_y^3 & 3d_x d_y^3 & d_y^4 & 4d_y^4 & 0 \\
e_x^4 & 0 & 2e_x^3 e_y & 0 & 3e_x^2 e_y^2 & 0 & 4e_x e_y^3 \\
f_x^3 f_y & 0 & 2f_x^2 f_y^2 & 0 & 3f_x f_y^3 & 0 & 4f_y^4 \\
\varphi_x^3 \varphi_y \psi_y & \varphi_x^2 \varphi_y^2 \psi_x & \varphi_x^2 \varphi_y^2 \psi_y & \varphi_x \varphi_y^3 \psi_x & \varphi_x \varphi_y^3 \psi_y & \varphi_y^4 \psi_x & \varphi_y^4 \psi_y
\end{matrix} \right|,$$

und dies ist umformbar auf

$$K = -2\,\triangle_y^4 \,.\, (x\,y)^6 \,.\, L,$$

wobei die Form L gegeben wird durch:

$$\begin{aligned}L = (cd)^2(ef)^2 c_y d_y . [&-12\, c_y\, d_y\, e_y^2\, f_y^2 \,.\, \varphi_x^3\, \varphi_y\, \psi_y + \\ &+ 36\, c_x\, d_y\, e_y^2\, f_y^2 \,.\, \varphi_x^2\, \varphi_y^2\, \psi_y + \\ &+ \{4\, c_y d_y\, (2\, e_x^2 f_y^2 + e_x e_y f_x f_y) - 48\, c_x d_y e_x f_y\, e_y f_y\} \varphi_x \varphi_y^3 \psi_y + \\ &+ \{12\, c_x d_y e_x f_y e_y f_y - 9\, c_x d_x e_y^2 f_y^2 - c_y d_y (2\, e_x^2 f_y^2 + e_x e_y f_x f_y)\}\, \varphi_y^4 \psi_x + \\ &+ \{18\, c_x d_y e_x e_y f_x f_y - 6\, e_x f_y c_y d_y e_x f_x\}\, \varphi_y^4\, \psi_y].\end{aligned}$$

Nun ist

$$\triangle_x^4 = (cd)^2\, c_x^2\, d_x^2 \quad D_{xy}\, \triangle_x^4 = 4\,(cd)^2\, c_x^2\, d_x\, d_y = 4\, \triangle_x^3\, \triangle_y$$

$$D_{xy}^2\, \triangle_x^4 = 4\,(cd)^2\, c_x^2\, d_y^2 + 8\,(cd)^2\, c_x\, c_y\, d_x\, d_y = 12\, \triangle_x^2\, \triangle_y^2.$$

Ferner ist

$$\begin{aligned}(cd)^2\, c_x\, c_y\, d_x\, d_y &= (cd)^2\, c_x^2\, d_y^2 - (cd)^2\, c_x\, d_y\, (cd)\, (xy) = \\ &= (cd)^2\, c_x^2\, d_y^2 - \tfrac{1}{2}\,(cd)^4\, (xy)^2 = (cd)^2\, c_x^2\, d_y^2 - \tfrac{1}{2}\, i\, (xy)^2.\end{aligned}$$

Daher wird

$$\left.\begin{aligned}&(cd)^2\, c_x^2\, d_y^2 = \triangle_x^2\, \triangle_y^2 + \tfrac{1}{3}\, i\, (xy)^2 \\ \text{und} \quad &(cd)^2\, c_x\, c_y\, d_x\, d_y = \triangle_x^2\, \triangle_y^2 - \tfrac{1}{6}\, i\, (xy)^2\end{aligned}\right\}.$$

Daher haben wir, wenn $\triangle$ und ϑ aequivalente Symbole sind:

$$\begin{aligned}L = &-12\, \vartheta_y^4\, \triangle_y^4\, \varphi_x^3\, \varphi_y\, \psi_y + 36\, \vartheta_y^4\, \triangle_y^3\, \triangle_x\, \varphi_x^2\, \varphi_y^2\, \psi_y + \\ &+ \{4\, \vartheta_y^4\, (3\, \triangle_x^2\, \triangle_y^2 + \tfrac{1}{2}\, i\, (xy)^2) - 48\, \vartheta_y^3\, \vartheta_x\, \triangle_y^3\, \triangle_x\}\, \varphi_x\, \varphi_y^3\, \psi_y + \\ &+ \{12\, \vartheta_y^3\, \vartheta_x\, \triangle_y^3\, \triangle_x - 9\, \vartheta_y^4\, (\triangle_x^2\, \triangle_y^2 - \tfrac{1}{6}\, i\, (xy)^2) - \vartheta_y^4\, (3\, \triangle_x^2\, \triangle_y^2 + \tfrac{1}{2}\, i\, (xy)^2)\}\, \varphi_y^4\, \psi_x + \\ &+ \{18\, \vartheta_y^3\, \vartheta_x\, (\triangle_x^2\, \triangle_y^2 - \tfrac{1}{6}\, i\, (xy)^2) - 6\, \vartheta_y^4\, \triangle_x^3\, \triangle_y\}\, \varphi_y^4\, \psi_y = \\ = &-12\, \vartheta_y^4\, \triangle_y^4\, \varphi_x^3\, \varphi_y\, \psi_y + 36\, \vartheta_y^4\, \triangle_y^3\, \triangle_x\, \varphi_x^2\, \varphi_y^2\, \psi_y + \\ &+ \{12\, \vartheta_y^4\, \triangle_x^2\, \triangle_y^2 - 48\, \vartheta_y^3\, \vartheta_x\, \triangle_y^3\, \triangle_x\}\, \varphi_x\, \varphi_y^3\, \psi_y + \\ &+ \{12\, \vartheta_y^3\, \vartheta_x\, \triangle_y^3\, \triangle_x - 12\, \vartheta_y^4\, \triangle_x^2\, \triangle_y^2\}\, \varphi_y^4\, \psi_x + \\ &+ \{18\, \vartheta_y^3\, \vartheta_x\, \triangle_x^2\, \triangle_y^2 - 6\, \vartheta_y^4\, \triangle_x^3\, \triangle_y\}\, \varphi_y^4\, \psi_y + \\ &+ i\, (xy)^2\, \{2\, \vartheta_y^4\, \varphi_x\, \varphi_y^3\, \psi_y + \vartheta_y^4\, \varphi_y^4\, \psi_x - 3\, \vartheta_y^3\, \vartheta_x\, \varphi_y^4\, \psi_y\} = L_1 + i\, (xy)^2\, L_2\end{aligned}$$

$$L_1 = -12\vartheta_y^4 \triangle_y^4 \varphi_x^3 \varphi_y \psi_y + 36\vartheta_y^4 \triangle_y^3 \triangle_x \varphi_x^2 \varphi_y^2 \psi_y +$$
$$+\{12\vartheta_y^4 \triangle_x^2 \triangle_y^2 - 48\vartheta_y^4 \triangle_y^2 \triangle_x^2 - 48\vartheta_y^3 \triangle_y^2 \triangle_x (\vartheta\triangle)(xy)\} \varphi_x \varphi_y^3 \psi_y +$$
$$+\{12\vartheta_y^4 \triangle_y^2 \triangle_x^2 + 12\vartheta_y^3 \triangle_y^2 \triangle_x (\vartheta\triangle)(xy) - 12\vartheta_y^4 \triangle_x^2 \triangle_y^2\} \varphi_y^4 \psi_x +$$
$$+\{18\vartheta_y^4 \triangle_x^3 \triangle_y + 18\vartheta_y^3 \triangle_x^2 \triangle_y (\vartheta\triangle)(xy) - 6\vartheta_y^4 \triangle_x^3 \triangle_y\} \varphi_y^4 \psi_y =$$
$$= -\{12\vartheta_y^4 \triangle_y \varphi_y \psi_y (\triangle_y^3 \varphi_x^3 - 3\triangle_y^2 \varphi_x^2 \triangle_x \varphi_y + 3\triangle_y \varphi_x \triangle_x^2 \varphi_y^2 - \triangle_x^3 \varphi_y^3) +$$
$$+6(xy)\varphi_y^3 \{-8(\vartheta\triangle)\vartheta_y^3 \triangle_y^2 \triangle_x \varphi_x \psi_y + 2(\vartheta\triangle)\vartheta_y^3 \triangle_y^2 \triangle_x \varphi_y \psi_x +$$
$$+3(\vartheta\triangle)\vartheta_y^3 \triangle_x^2 \triangle_y \varphi_y \psi_y\}.$$

Nun ist weiter

$$(\vartheta\triangle)\vartheta_y^3 \triangle_y^2 \triangle_x = \tfrac{1}{2}(\vartheta\triangle)\vartheta_y^2 \triangle_y^2 (\vartheta_y \triangle_x - \vartheta_x \triangle_y) = -\tfrac{1}{2}(\vartheta\triangle)^2 \vartheta_y^2 \triangle_y^2 (xy)$$

und

$$(\vartheta\triangle)\vartheta_y^3 \triangle_y \triangle_x^2 = \tfrac{1}{2}(\vartheta\triangle)\vartheta_y \triangle_y (\vartheta_y^2 \triangle_x^2 - \vartheta_x^2 \triangle_y^2) = -(\vartheta\triangle)^2 \vartheta_y^2 \triangle_x \triangle_y (xy).$$

Daher wird:

$$L_1 = -12\vartheta_y^4 \triangle_y \varphi_y \psi_y (\triangle_y \varphi_x - \triangle_x \varphi_y)^3 +$$
$$+6(xy)^2 \varphi_y^3 (\vartheta\triangle)^2 \vartheta_y^2 \triangle_y \{4\triangle_y \varphi_x \psi_y - \triangle_y \varphi_y \psi_x - 3\triangle_x \varphi_y \psi_y\} =$$
$$= +12(xy)^3 \vartheta_y^4 \triangle_y \varphi_y \psi_y (\triangle\varphi)^3 + 6(xy)^2 (\vartheta\triangle)^2 \vartheta_y^2 \triangle_y$$
$$\{4\triangle_y (\varphi\psi)(xy) + 3\varphi_y (\triangle\psi)(yx)\} \varphi_y^3 =$$
$$= (xy)^3 \{12\vartheta_y^4 (\triangle\varphi)^3 \triangle_y \varphi_y \psi_y + 24(\vartheta\triangle)^2 \vartheta_y^2 \triangle_y^2 \varphi_y^3 (\varphi\psi) -$$
$$-18(\vartheta\triangle)^2 \vartheta_y^2 \triangle_y (\triangle\psi) \varphi_y^4\}.$$

Ferner ist

$$L_2 = 2\vartheta_y^4 \psi_x \varphi_y^3 \psi_y + \vartheta_y^4 \varphi_y^4 \psi_x - 3\vartheta_y^3 \vartheta_x \varphi_y^4 \psi_y =$$
$$= 2\vartheta_y^4 \varphi_y^3 (\varphi\psi)(xy) + 3\vartheta_y^3 \varphi_y^4 (\vartheta\psi)(yx) = (xy)\{2\vartheta_y^4 \varphi_y^3 (\varphi\psi) - 3\vartheta_y^3 \varphi_y^4 (\vartheta\psi)\}.$$

Also kommt schliesslich:

$$L = (xy)^3 . [12\vartheta_y^4 (\triangle\varphi)^3 \triangle_y \varphi_y \psi_y + 24(\vartheta\triangle)^2 \vartheta_y^2 \triangle_y^2 \varphi_y^3 (\varphi\psi) -$$
$$-18(\vartheta\triangle)^2 \vartheta_y^2 \triangle_y (\triangle\psi) \varphi_y^4 + i\{2\vartheta_y^4 \varphi_y^3 (\varphi\psi) - 3\vartheta_y^3 \varphi_y^4 (\vartheta\psi)\}].$$

Es kommt also an auf die Form siebenter Stufe:

$$M = 12\vartheta_y^4 (\triangle\varphi)^3 \triangle_y \varphi_y \psi_y + 24(\vartheta\triangle)^2 \vartheta_y^2 \triangle_y^2 \varphi_y^3 (\varphi\psi) -$$
$$-18(\vartheta\triangle)^2 \vartheta_y^2 \triangle_y (\triangle\psi) \varphi_y^4 + i\{2\vartheta_y^4 \varphi_y^3 (\varphi\psi) - 3\vartheta_y^3 \varphi_y^4 (\vartheta\psi)\},$$

wobei

$$K = -2(xy)^6 \triangle_y^4 L = -2(xy)^9 . \triangle_y^4 . M.$$

Setzt man hierin $x_1 = 1$ $x_2 = 0$ $y_1 = 0$ und $y_2 = 1$, so erhält man aus M den Koeffizienten M_7 von y_2^7; aus K bekommt man aber durch dieselbe Spezialisierung die Determinante der letzten sieben Kolonnen aus der Matrix m.

Ebenso wie diese enthält M_7 keine Γ-Symbole.

Daher ist

$$M_7\,(a_{iklm},\, \varphi_{rstu}\,\psi_v) = M_7\left(a_{iklm},\, \frac{\partial a_{rstu}}{\partial x_v}\right).$$

Den Koeffizienten $M_6\,(a, \varphi\,\psi)$ aus M erhält man aus $M_7\,(a, \varphi\,\psi)$ durch Anwendung des binären Prozesses $\triangle_{21}$ [5]). Nun ist

$$M_7\,(a_{iklm},\, \varphi_{rstu}\,\psi_v) \equiv M_7\left(a_{iklm},\, \frac{\partial a_{rstu}}{\partial x_v}\right) - M_7\,(a_{iklm},\, d_{rstu,\,v}).$$

Da

$$M_7\,(a_{iklm},\, d_{rstu,\,v}) \equiv 0\ \{a, \Gamma\}$$

und $\triangle_{21}$ einer linearen Transformation entspricht, für welche die $d_{rstu,v}$ sich wie die Komponenten eines kovarianten Tensors verhalten, ist auch

$$\triangle_{21}\, M_7\,(a_{iklm},\, d_{rstu,\,v}) \equiv 0\ \{a, \Gamma\}.$$

Somit wird

$$M_6\,(a_{iklm},\, \varphi_{rstu}\,\psi_v) \equiv M_6\left(a_{iklm},\, \frac{\partial a_{rstu}}{\partial x_v}\right).$$

Auf dieselbe Art zeigt man, dass auch die übrigen Koeffizienten von M keine Γ-Symbole enthalten, d.h. in M kann man $\varphi_{rstu}\,\psi_v$ durch $\dfrac{\partial a_{rstu}}{\partial x_v}$ ersetzen.

§ 6. Der obigen Form M kann man nun eine andere Gestalt geben. Setzt man [4])

$$(\vartheta\,\triangle)^2\,\vartheta_y^2\,\triangle_y^2 = \tfrac{1}{3}\,j\,f - \tfrac{1}{6}\,i\,\triangle = E$$

so ist

$$(\vartheta\,\triangle)^2\,\vartheta_y^2\,\triangle_y\,\triangle_1 = \tfrac{1}{4}\,\frac{\partial E}{\partial y_1}$$

also

$$M = 12\,\triangle\,(\triangle\,\varphi)^3\,\triangle_y\,\varphi_y\,\psi_y + 24\,E\,\varphi_y^3(\varphi\psi) - \tfrac{9}{2}\left(\frac{\partial E}{\partial y}\,\psi\right)\varphi_y^4 +$$

$$+\, i\left\{2\,\triangle\,\varphi_y^3\,(\varphi\,\psi) - \tfrac{3}{4}\left(\frac{\partial\triangle}{\partial y}\,\psi\right)\varphi_y^4\right\} =$$

5) Vgl. R. WEITZENBÖCK, Invariantentheorie, S. 220.

$$=12\,\triangle\,(\triangle\,\varphi)^3\,\triangle_y\,\varphi_y\,\psi_y+(8\,j\,f-4\,i\,\triangle)\,\varphi_y^3\,(\varphi\,\psi)-$$

$$-\left\{\tfrac{3}{2}\,j\left(\frac{\partial f}{\partial y}\,\psi\right)-\tfrac{3}{4}\,i\left(\frac{\partial\triangle}{\partial y}\,\psi\right)\right\}\varphi_y^4+2\,i\,\triangle\,\varphi_y^3(\varphi\psi)-\tfrac{3}{4}\,i\left(\frac{\partial\triangle}{\partial y}\,\psi\right)\varphi_y^4=$$

$$=12\,\triangle\,.(\triangle\,\varphi)^3\,\triangle_y\,\varphi_y\,\psi_y-\tfrac{3}{2}\,j\left(\frac{\partial f}{\partial y}\,\psi\right)\varphi_y^4+(8\,j\,f-2\,i\,\triangle)\,\varphi_y^3\,(\varphi\psi).$$

Jetzt kann man $\varphi_{rstu}\,\psi_v$ durch $\dfrac{\partial\,a_{rstu}}{\partial\,x_v}$ ersetzen. Man findet dann, wenn man für y_1 und y_2 wieder dx^1 und dx^2 schreibt, und noch

$$\frac{\partial a_{iklm}}{\partial x_1}\,dx^i\,dx^k\,dx^l\,dx^m=g_1 \quad \text{und} \quad \frac{\partial a_{iklm}}{\partial x_2}\,dx^i\,dx^k\,dx^l\,dx^m=g_2$$

setzt:

$$(\triangle\,\varphi)^3\,\triangle_y\,\varphi_y\,\psi_y=(\triangle,g_1)^{(3)}\,dx^1+(\triangle,g_2)^{(3)}\,dx^2$$

worin $(\triangle,g_1)^{(3)}$ die dritte Ueberschiebung von $\triangle$ und g_1 bedeutet.

Es ist überdies:

$$\left(\frac{\partial f}{\partial y}\,\psi\right)\varphi_y^4=\begin{vmatrix}\dfrac{\partial f}{\partial y_1} & \dfrac{\partial f}{\partial y_2}\\ \varphi_y^4\,\psi_1 & \varphi_y^4\,\psi_2\end{vmatrix}=\begin{vmatrix}\dfrac{\partial f}{\partial dx^1} & \dfrac{\partial f}{\partial dx^2}\\ g_1 & g_2\end{vmatrix}$$

$$\varphi_y^3\,(\varphi\,\psi)=\begin{vmatrix}\varphi_y^3\,\varphi_1 & \varphi_y^3\,\varphi_2\\ \psi_1 & \psi_2\end{vmatrix}=\tfrac{1}{4}\left(\frac{\partial g_2}{\partial dx^1}-\frac{\partial g_1}{\partial dx^2}\right).$$

Die binäre biquadratische Differentialform hat also eine Differentialkovariante erster Ordnung und siebenter Stufe:

$$M=12\,\triangle\,\{(\triangle,g_1)^{(3)}\,dx^1+(\triangle,g_2)^{(3)}\,dx^2\}-\tfrac{3}{2}\,j\left\{g_2\,\frac{\partial f}{\partial dx^1}-g_1\,\frac{\partial f}{\partial dx^2}\right\}+$$

$$+(2\,j\,f-\tfrac{1}{2}\,i\,\triangle)\left(\frac{\partial g_2}{\partial dx^1}-\frac{\partial g_1}{\partial dx^2}\right).$$

Mathematics. — *On Waring's problem.* By M. M. FULD. (Communicated by Prof. W. VAN DER WOUDE).

(Communicated at the meeting of February 26, 1938.)

H. DAVENPORT and H. HEILBRONN proved in 1936 [1]), that every sufficiently large integer is a sum of 17 fourth powers [2]), while before the best known result was that every sufficiently large integer is a sum of 19 biquadrates [3]). One may ask, whether the method followed by DAVENPORT-HEILBRONN-ESTERMANN also gives better results for higher than 4^{th} powers, and the following results have been obtained by the author:

Theorem 1 [4]).

Let k be an integer $\geqq 4$, and $G(k)$ the smallest of all numbers g, such that every sufficiently large integer is a sum of g k^{th} powers of non-negative integers. If then the integers s and j satisfy the inequalities

$$\frac{7}{5}\,2^{k-1} \geqq s > 2k, \quad k-1 \leqq j < k^2, \quad \left(\frac{k-1}{k}\right)^j < \frac{s-2}{2^{k-1}(k-2)},$$

then

$$G(k) \leqq s + 2j + 2.$$

Particularly (in brackets the best earlier results are given):

$G(4) \leqq 17$ (17), $G(5) \leqq 29$ (41), $G(6) \leqq 42$ (87), $G(7) \leqq 59$ (137), $G(8) \leqq 78$ (163), $G(9) \leqq 101$ (189), $G(10) \leqq 125$ (216), $G(11) \leqq 153$ (243), $G(12) \leqq 184$ (271), $G(13) \leqq 217$ (299), $G(14) \leqq 253$ (328), $G(15) \leqq 291$ (358), $G(16) \leqq 333$ (387), $G(17) \leqq 376$ (418), $G(18) \leqq 425$ (448), $G(19) \leqq 474$ (479), $G(20) \leqq 529$ (510) [5]).

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1) H. DAVENPORT and H. HEILBRONN, "On Waring's problem for fourth powers". Proceedings of the London Mathematical Society, (2) **41**, 143—150 (1936).

2) At the same time this result has also been proved by T. ESTERMANN, "Proof that every large integer is a sum of seventeen biquadrates". Proceedings of the London Mathematical Society, (2) **41**, 126—142 (1936).

3) G. H. HARDY and J. E. LITTLEWOOD, "Some problems of "Partitio Numerorum", 6", "Further researches in Waring's problem". Mathematische Zeitschrift, **23**, 1—37 (1925).

4) In Acta Arithmetica, **2**, 197—211 (1937), ESTERMANN just now obtained a formula which gives the same new results as theorem 1, with the same method, he followed before.

5) The earlier results for $k=4$ are due to DAVENPORT–HEILBRONN and ESTERMANN, for $k=5$ and $k=6$ to HARDY–LITTLEWOOD and for the larger k to HEILBRONN.

For $k \geqq 20$ the above theorem gives worse results than HEILBRONN's formula

$$G(k) \leqq 6\,k\,\log k + k\left(4 + 3\log\left(3 + \frac{2}{k}\right)\right) + 3\ ^{6)}.$$

This formula will be improved by theorem 2.

Theorem 2.

If k is an integer $\geqq 4$, s, j and l are integers, which satisfy the inequalities

$$\frac{7}{5}\,2^{k-1} \geqq s > 2k, \quad k-1 \leqq j < k^2, \quad l < k^2$$

$$\tfrac{1}{2}\left(1 - \frac{2k-1}{k(2k+1)}\right)\left(\frac{k-1}{k}\right)^{l-2} + \\ + \left(\frac{k-1}{k}\right)^{j} < \tfrac{1}{2}\,\frac{2k-1}{k(k-2)(2k+1)} + \frac{(2k-1)(s-2)}{(2k+1)(k-2)\,2^{k-1}},$$

then

$$G(k) \leqq s + 2j + l + 2.$$

From this theorem it follows, that

$$G(k) \leqq 6\,k\,\log k + k\,(2 + 3\log 3) + 8.$$

Particularly theorem 2 gives :

$$G(17) \leqq 369, \quad G(18) \leqq 398, \quad G(19) \leqq 427, \quad G(20) \leqq 455,$$

which results are better than those given by theorem 1.

In proving these theorems the same method has been applied, as DAVENPORT-HEILBRONN [1]) follow for $k = 4$. For the proof of theorem 1 a result of ESTERMANN [2]) is used and for theorem 2 a function, used by HEILBRONN [6]) is adopted.

The improvement on the result of HEILBRONN has been obtained by applying a better result concerning the singular series.

The proof could somewhat be simplified by substituting for the auxiliary-function used by DAVENPORT-HEILBRONN a form which does not contain the Γ-function ; in consequence of this the Γ-function does not occur any more in the whole proof.

The proofs will be published in my dissertation.

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[6]) H. HEILBRONN, "Ueber das Waringsche Problem". Acta Arithmetica, **1**, 212—221 (1936).

Mathematics. — *Quelques applications de la méthode des approximations successives.* Par H. BREMEKAMP. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of February 26, 1938.)

§ 1. La méthode des approximations successives dans la théorie des équations différentielles, publiée en 1890 par M. E. PICARD a été employée principalement pour des recherches d'un caractère théorique, comme la démonstration de l'existence et de l'unicité de la solution de l'équation sous des conditions convenables. Cependant, il me semble, la méthode a aussi une certaine valeur pour la solution effective de plusieurs équations aux dérivées partielles, qu'on rencontre dans les applications.

Dans ces cas, la détermination du domaine de validité des solutions est une question de grand intérêt. M. PICARD s'est borné dans la plupart des cas à des démonstrations valables dans un domaine suffisamment petit, sans déterminer ce domaine d'une manière plus précise. Pour le cas de l'équation du second ordre du type hyperbolique, complètement linéaire,

$$\frac{\partial^2 z}{\partial x\,\partial y} = a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz + f,$$

où a, b, c, f, sont des fonctions continues de x et y, M. E. GOURSAT, en modifiant un peu la méthode de raisonnement, a démontré, que la méthode de PICARD nous permet, d'obtenir la solution, qui prend des valeurs données sur deux portions de caractéristiques OA et OB, dans tout le rectangle $OACB$.

Les résultats pour les équations analogues, mais non complètement linéaires ne sont pas si beaux, mais encore assez intéressants pour attirer l'attention des mathématiciens et des ingénieurs sur ce point.

§ 2. Comme premier exemple je vais m'occuper d'un système d'équations non complètement linéaires qu'on trouve dans le rapport de la commission, qui a étudié les conséquences possibles du barrage de la Zuiderzee.

Le mouvement de l'eau dans un des goulets considérés par la commission est assujetti aux équations

$$\frac{\partial s}{\partial x} = -b\frac{\partial h}{\partial t}$$

$$\frac{\partial s}{\partial t} = -bgq\frac{\partial h}{\partial x} \mp \frac{bq}{\varrho}W + \frac{bq}{\varrho}F$$

L'axe des x est mené par l'axe du goulet, t représente le temps, s le courant d'eau à travers la section entière du goulet, h la hauteur de l'eau au dessus d'un certain plan de niveau V, choisi de sorte que par les mouvements considérés la surface de l'eau ne s'écarte que peu du plan V, q la distance de V au fond du goulet, b la largeur du goulet, g la constante de la gravitation et ϱ la densité de l'eau. Enfin W représente la résistance du sol et des parois et F la force extérieure, toutes les deux par unité de volume. Dans les considérations sur les marées la dernière peut être négligée. La résistance W est toujours opposée à la vitesse, c'est pour ça que le terme contenant W a les deux signes, le signe supérieur est valable pour s positif, le signe inférieur pour s négatif. La valeur absolue de W est $\frac{g\varrho}{q^3 C^2} s^2$, où C est une certaine constante (constante d'Eytelwein). Il faut encore remarquer, que h est considéré comme petit envers q. Pour plus de détails sur la signification des symboles, nous renvoyons au rapport de la commission.

La commission a évité de s'occuper des difficultés, qui résultent du caractère quadratique du terme représentant la résistance, en posant $W = ks$, et en déterminant le coëfficient k, suivant un procédé indiqué par son président, LORENTZ, tel que les mouvements calculés s'écartent aussi peu que possible du mouvement réel. Nous allons résoudre ces équations en retenant le terme quadratique.

En précisant nous cherchons la solution du système d'équations

$$\frac{\partial s}{\partial x} = -b\frac{\partial h}{\partial t}, \quad \ldots \ldots \ldots \ldots \quad (1)$$

$$\frac{\partial s}{\partial t} \pm \frac{bg}{q^2C^2} s^2 = -bgq\frac{\partial h}{\partial x}, \quad \ldots \ldots \ldots \quad (2)$$

où pour $x = 0$, h et s sont égaux à des fonctions de t données.

On pourrait éviter le signe double dans l'équation (2) en posant pour le terme en question $\frac{bg}{q^2C^2} s|s|$. Ça ne change prèsque rien aux considérations qui suivent. Nous nous bornerons au cas que s reste positif. Alors vaut dans (2) le signe $+$. De plus nous nous tiendrons au cas que b et q sont constantes. Il n'y a d'ailleurs pas beaucoup à changer à la méthode, si ces quantités sont fonctions de x, ce qui pourrait être d'une certaine importance dans les applications. En éliminant h de (1) et (2) nous trouvons

$$\frac{\partial^2 s}{\partial t^2} - gq\frac{\partial^2 s}{\partial x^2} = -\frac{2bg}{C^2q^2} s\frac{\partial s}{\partial t}$$

et les conditions aux limites pourront être mises sous la forme: pour

$x=0,\ s=\varphi_1(t), \frac{\partial s}{\partial x}=\varphi_2(t)$. Nous posons $g\,q=w^2,\ \frac{bg}{C^2 q^2}=2\,aw$, puis nous substituons $x+wt=\xi, x-wt=\eta$, c'est-à-dire nous rapportons l'équation à ses caractéristiques. Il vient alors

$$\frac{\partial^2 s}{\partial\xi\,\partial\eta}=a\,s\left(\frac{\partial s}{\partial\xi}-\frac{\partial s}{\partial\eta}\right)$$

et les conditions aux limites deviennent:

$$\text{pour } \xi+\eta=0,\quad s=f_1(\xi),\quad \frac{\partial s}{\partial\xi}=f_2(\xi),$$

nous pouvons encore les mettre sous une forme plus symmétrique, comme il suit:

$$\text{pour } \xi=\eta=0,\ s=s_0,\ \text{puis pour } \xi+\eta=0,\ \frac{\partial s}{\partial\xi}=g_1(\xi),\ \frac{\partial s}{\partial\eta}=g_2(\xi).$$

§ 3. Nous résoudrons un problème un peu plus général et nous lâcherons dans ce qui suit les notations qui se rapportent à l'application spéciale que nous avions en vue d'abord.

De l'équation

$$\frac{\partial^2 z}{\partial x\,\partial y}=a\,z\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right), \quad \ldots\ldots\ldots \quad (3)$$

où a est une fonction continue donnée de x et y, nous cherchons une solution telle qu'aux points d'une certaine courbe donnée k, z et $\frac{\partial z}{\partial n}$ (n indiquant la direction de la normale au point considéré) prennent des valeurs données d'avance. Nous supposons que la courbe donnée a en chaque point une tangente déterminée qui en aucun point de la portion de courbe considérée n'est parallèle à l'un des axes de coördonnées.

Ces conditions aux limites peuvent s'exprimer comme il suit.

Soient les équations de la courbe k

$$x=\varphi_1(u),\quad y=\varphi_2(u),\quad a<u<b \quad \ldots\ldots \quad (4)$$

φ_1 et φ_2 étant dans l'intervalle considéré des fonctions ayant une dérivée à signe constant, (admettons, pour fixer les idées, φ_1' positif et φ_2' négatif).

Sur la courbe on aura

$$\frac{\partial z}{\partial x}=\psi_1(u),\quad \frac{\partial z}{\partial y}=\psi_2(u) \quad \ldots\ldots\ldots \quad (5)$$

et pour $u=0$, $z=c$.

La méthode des approximations successives consiste en ceci. Nous

déterminons d'abord la fonction z_0, satisfaisant aux conditions données sur la courbe k et à l'équation

$$\frac{\partial^2 z_0}{\partial x \, \partial y} = 0 \ , \quad \ldots \ldots \ldots \ldots \quad (6)$$

puis la fonction z_1, satisfaisant aux conditions sur la courbe et à l'équation

$$\frac{\partial^2 z_1}{\partial x \, \partial y} = a \, z_0 \left(\frac{\partial z_0}{\partial x} - \frac{\partial z_0}{\partial y} \right),$$

et ainsi de suite, pour $n \geqq 1$

$$\frac{\partial^2 z_n}{\partial x \, \partial y} = a \, z_{n-1} \left(\frac{\partial z_{n-1}}{\partial x} - \frac{\partial z_{n-1}}{\partial y} \right),$$

toutes les fonctions z_n satisfaisant aux conditions sur la courbe k.

Nous allons montrer que ces fonctions convergent vers une fonction limite z et que cette fonction

$$z = \lim_{n \to \infty} z_n$$

satisfait à toutes les conditions du problème.

Posons $z_0 = v_0$, $z_n - z_{n-1} = v_n$ $(n \geqq 1)$ par suite

$$z_n = v_0 + v_1 + v_2 + \ldots . v_n .$$

Nous avons à démontrer en premier lieu, que la série $\Sigma \, v_n$ est convergente dans une certaine bande autour de la courbe k.

Les fonctions v_k sont déterminées par les données suivantes.

Aux points de la courbe k,

$$\frac{\partial v_0}{\partial x} = \psi_1(u), \ \frac{\partial v_0}{\partial y} = \psi_2(u),$$

et pour

$$u = 0, \ v_0 = c,$$

puis pour

$$k \geqq 1, \ v_k = 0, \ \frac{\partial v_k}{\partial x} = 0, \ \frac{\partial v_k}{\partial y} = 0.$$

De plus

$$\frac{\partial^2 v_0}{\partial x \, \partial y} = 0,$$

$$\frac{\partial^2 v_1}{\partial x \, \partial y} = a \, v_0 \left(\frac{\partial v_0}{\partial x} - \frac{\partial v_0}{\partial y} \right),$$

et pour $k > 1$

$$\frac{\partial^2 v_k}{\partial x \, \partial y} = a \left\{ z_{k-1} \left(\frac{\partial z_{k-1}}{\partial x} - \frac{\partial z_{k-1}}{\partial y} \right) - z_{k-2} \left(\frac{\partial z_{k-2}}{\partial x} - \frac{\partial z_{k-2}}{\partial y} \right) \right\}$$

$$= a \left\{ v_{k-1} \frac{\partial z_{k-1}}{\partial x} + z_{k-2} \frac{\partial v_{k-1}}{\partial x} - v_{k-1} \frac{\partial z_{k-1}}{\partial y} - z_{k-2} \frac{\partial v_{k-1}}{\partial y} \right\},$$

donc

$$\frac{\partial^2 v_k}{\partial x\,\partial y}=a\left\{v_{k-1}\sum_0^{k-1}\frac{\partial v_n}{\partial x}+\frac{\partial v_{k-1}}{\partial x}\sum_0^{k-2}v_n-v_{k-1}\sum_0^{k-1}\frac{\partial v_n}{\partial y}-\frac{\partial v_{k-1}}{\partial y}\sum_0^{k-2}v_n\right\}. \quad . \quad (8$$

Il s'en suit (voir la fig. 1, où A, B et C sont les points caractérisés par les valeurs du paramètre $u=a$, $u=b$, $u=0$ respectivement)

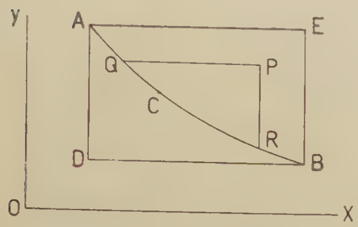

Fig. 1

$$v_{0_P}=v_{0_Q}+\int_Q^R v_{0_S}\frac{\partial x}{\partial s}\,ds,$$

ou encore

$$v_{0_P}=c+\int_0^{u_R}\psi_1\,\varphi_1'\,du+\int_0^{u_Q}\psi_2\,\varphi_2'\,du.$$

En général pour une fonction v satisfaisant à l'équation

$$\frac{\partial^2 v}{\partial x\,\partial y}=f(x,y),$$

et se réduisant à zéro avec ses dérivées premières sur la courbe k nous avons

$$v_P=\iint f(x,y)\,dx\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

l'intégrale double étant étendue à l'aire du triangle curviligne QPR,

$$\frac{\partial v_P}{\partial x}=\int_R^P f(x_P,y)\,dy,\quad \frac{\partial v_P}{\partial y}=\int_Q^P f(x,y_P)\,dx. \quad . \quad . \quad . \quad . \quad (10)$$

Par nos données les fonctions v_k sont déterminées à l'intérieur du rectangle $ADBE$. Nous choississons pour simplifier DB et DA comme axes des coördonnées et pour fixer les idées nous plaçons le point P dans le triangle ABE.

Soit dans ce domaine

$$|a|<m,\ \left|\frac{\partial v_0}{\partial x}\right|<M,\ \left|\frac{\partial v_0}{\partial y}\right|<M,\ |v_0|<M(x+y).$$

Nous allons montrer, que, tant que

$$2m(x+y)e^{M(x+y)}\leqq 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\left|\frac{\partial v_k}{\partial x}\right|<\frac{M^{k+1}(x+y)^k}{k!},\ \left|\frac{\partial v_k}{\partial y}\right|<\frac{M^{k+1}(x+y)^k}{k!} \quad . \quad . \quad . \quad (12)$$

et par suite

$$|v_k|<\frac{M^{k+1}(x+y)^{k+1}}{(k+1)!}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Si ces inégalités tiennent pour $k = 0, 1, 2 \ldots (n-1)$, nous déduisons de (8) et (10)

$$\left|\frac{\partial v_n}{\partial x}\right| < 2m \int_R^P \left\{ \frac{M^n (x+y)^n}{n!} \sum_0^{n-1} \frac{M^{k+1}(x+y)^k}{k!} + \right.$$

$$\left. + \frac{M^n (x+y)^{n-1}}{(n-1)!} \sum_0^{n-2} \frac{M^{k+1}(x+y)^{k+1}}{(k+1)!} \right\} dy$$

$$< 2m \frac{M^n}{n!} \sum_0^{\infty} \frac{M^{k+1}(x+y)^{n+k+1}}{k!\,(n+k+1)} +$$

$$+ 2m \frac{M^n}{(n-1)!} \sum_0^{\infty} \frac{M^{k+1}(x+y)^{n+k+1}}{(k+1)!\,(n+k+1)}$$

$$< 2m \frac{M^{n+1}(x+y)^n}{n!} (x+y)\, e^{M(x+y)} \left(\frac{1}{n+1} + \frac{n}{n+1}\right) <$$

$$< \frac{M^{n+1}(x+y)^n}{n!}$$

de même pour $\left|\dfrac{\partial v_n}{\partial y}\right|$.

Les inégalités tiennent pour $k = 0$, elles tiennent donc pour toutes les valeurs entières positives de k.

De l'inégalité (13) nous concluons que la série $\Sigma\, v_n$ est convergente et par suite que $\lim\limits_{n \to \infty} z_n$ existe. Soit $\lim\limits_{n \to \infty} z_n = z$. Il est clair, que z satisfait aux conditions données sur la courbe k. Nous allons montrer, que z satisfait aussi à l'équation (3).

Il suit de (13), que la série $\Sigma \dfrac{\partial v_k}{\partial x}$ est uniformément convergente dans un domaine quelconque intérieur à la bande définie à l'aide de (11), par suite

$$\frac{\partial}{\partial x} \Sigma\, v_k = \Sigma \frac{\partial v_k}{\partial x},$$

ou

$$\frac{\partial z}{\partial x} = \lim_{n \to \infty} \frac{\partial z_n}{\partial x}$$

de même

$$\frac{\partial z}{\partial y} = \lim_{n \to \infty} \frac{\partial z_n}{\partial y}.$$

Faisons alors augmenter n indéfiniment dans l'équation (7); il est clair, que le second membre converge vers $\alpha z \left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$. Quant au premier

membre nous, pouvons l'écrire $\sum\limits_{1}^{n} \frac{\partial^2 v_k}{\partial x\, \partial y}$. Démontrons, que $\Sigma \frac{\partial^2 v_k}{\partial x\, \partial y}$ est une série convergente. De (8), (12) et (13) nous tirons

$$\left| \frac{\partial^2 v_k}{\partial x \partial y} \right| < 2\, m \left\{ \left| v_{k-1} \right| M + \left| \frac{\partial v_{k-1}}{\partial x} \right| \right\} e^{M(x+y)},$$

d'où nous concluons en appliquant de nouveau (12) et (13) que $\Sigma \frac{\partial^2 v_k}{\partial x\, \partial y}$ converge uniformément dans tout domaine intérieur à la bande considérée. Si nous posons $\frac{\partial v_k}{\partial x} = U_k$, de sorte que le premier membre de (8) est $\frac{\partial U_k}{\partial y}$, il suit de cette convergence uniforme

$$\Sigma \frac{\partial^2 v_k}{\partial x \partial y} = \Sigma \frac{\partial U_k}{\partial y} = \frac{\partial}{\partial y} \Sigma U_k = \frac{\partial}{\partial y} \Sigma \frac{\partial v_k}{\partial x} = \frac{\partial^2 z}{\partial x \partial y},$$

où pour la dernière conclusion nous nous appuyons sur la convergence uniforme de $\Sigma \frac{\partial v_k}{\partial x}$, déjà démontrée. Nous avons donc

$$\frac{\partial^2 z}{\partial x \partial y} = a\, z \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right).$$

D'une manière tout semblable on démontre, que pour

$$2\, M(x+y)\, e^{m(x+y)} \leqq 1,$$

$$\left| \frac{\partial v_k}{\partial n} \right| < \frac{m^k (x+y)^k}{k!} M, \quad \left| \frac{\partial v_k}{\partial y} \right| < \frac{m^k (x+y)^k}{k!} M, \quad \left| v_k \right| < \frac{m^k (x+y)^{k+1}}{(k+1)!} M,$$

ce qui permet les mêmes conclusions.

§ 4. En modifiant un peu la démonstration on trouve une limite inférieure pour la largeur de la bande, qui dans certains cas pourra être plus grande.

Nous allons démontrer, que pour

$$m(x+y) < 1, \quad M(x+y) < 1 \quad \text{et} \quad 2\, m(x+y) \lg \frac{1}{1 - M(x+y)} \leqq 1, \quad . \quad . \quad . \tag{14}$$

nous aurons

$$\left| \frac{\partial v_k}{\partial x} \right| < M^{k+1} (x+y)^k, \quad \left| \frac{\partial v_k}{\partial y} \right| < M^{k+1} (x+y)^k, \quad . \quad . \quad . \quad . \tag{15}$$

et par suite

$$\left| v_k \right| < \frac{M^{k+1} (x+y)^{k+1}}{k+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{16}$$

Ces inégalités tiennent pour $k=0$. Puis nous avons

$$\frac{\partial v_1}{\partial x}=\int_R^P a v_0\left(\frac{\partial v_0}{\partial x}-\frac{\partial v_0}{\partial y}\right)\partial y,$$

par suite

$$\left|\frac{\partial v_1}{\partial x}\right|<2mM^2\int_R^P(x+y)\,dy<mM^2(x+y)^2<M(x+y),$$

de même $\left|\frac{\partial v_1}{\partial y}\right|<M(x+y)$.

Admettons, que (15) et (16) sont valables pour $k=0,\ 1,\ 2,\dots n-1$. Nous avons alors pour $n\geqq 2$

$$\left|\frac{\partial v_n}{\partial x}\right|<2m\int_R^P\left\{\frac{M^n(x+y)^n}{n}\sum_0^{n-1}M^{k+1}(x+y)^k+\right.$$

$$\left.+M^n(x+y)^{n-1}\sum_0^{n-2}\frac{M^{k+1}(x+y)^{k+1}}{k+1}\right\}dy$$

$$<2m\left\{\frac{M^n(x+y)^n}{n}\sum_0^{\infty}\frac{M^{k+1}(x+y)^{k+1}}{n+k+1}+\right.$$

$$\left.+M^n(x+y)^n\sum_0^{\infty}\frac{M^{k+1}(x+y)^{k+1}}{(k+1)(n+k+1)}\right\},$$

d'où, parce que

$$\frac{1}{(k+1)(n+k+1)}=\frac{1}{n}\left\{\frac{1}{k+1}-\frac{1}{n+k+1}\right\},$$

$$\left|\frac{\partial v_n}{\partial x}\right|<\frac{4m}{n}M^{n+1}(x+y)^{n+1}\,lg\,\frac{1}{1-M(x+y)}<M^{n+1}(x+y)^n,$$

car
$$\frac{2}{n}\leqq 1 \text{ et } 2m\,lg\,\frac{1}{1-M(x+y)}\leqq 1.$$

De même $\left|\frac{\partial v_n}{\partial y}\right|<M^{n+1}(x+y)^n$.

Les inégalités (15) et (16) sont donc valables pour toutes les valeurs entières positives de k.

On en conclut comme au fin du § précédent, que le procédé des approximations successives conduit à la solution du problème.

On arrive à la même conclusion, si au lieu de (14) on a

$$m(x+y)<1,\ M(x+y)<1,\ 2M(x+y)\,lg\,\frac{1}{1-m(x+y)}<1. \quad (17)$$

Remarquons encore que les trois inégalités (13) et de même (17) sont comprises dans les deux suivantes

$$2m(x+y)<1, \quad 2M(x+y)<1.$$

§ 5. Comme second exemple considérons l'équation

$$\frac{\partial^2 z}{\partial x\,\partial y}=\alpha\left(\frac{\partial z}{\partial x}\right)^2+\beta\left(\frac{\partial z}{\partial y}\right)^2$$

α et β étant des fonctions continues de x et y et cherchons la solution satisfaisant aux conditions sur la courbe k, données précédemment. Employant des notations analogues à celles, qui sont introduites dans l'exemple précédent, nous avons

$$\frac{\partial^2 v_k}{\partial x\,\partial y}=\alpha\frac{\partial v_{k-1}}{\partial x}\left\{2\sum_0^{k-2}\frac{\partial v_n}{\partial x}+\frac{\partial v_{k-1}}{\partial x}\right\}+\beta\frac{\partial v_{k-1}}{\partial y}\left\{2\sum_0^{k-2}\frac{\partial v_n}{\partial y}+\frac{\partial v_{k-1}}{\partial y}\right\}. \tag{18}$$

Supposons, que dans le domaine considéré

$$|\alpha|<m, \quad |\beta|<m, \quad \left|\frac{\partial v_0}{\partial x}\right|<M, \quad \left|\frac{\partial v_0}{\partial y}\right|<M.$$

Nous pouvons démontrer que, pour

$$4me^{M(x+y)}\leqq 1, \quad \ldots\ldots\ldots \tag{19}$$

$$\left|\frac{\partial v_k}{\partial x}\right|<\frac{M^{k+1}(x+y)^k}{k!} \quad \text{et} \quad \left|\frac{\partial v_k}{\partial y}\right|>\frac{M^{k+1}(x+y)^k}{k!}.$$

Quand ces inégalités tiennent pour $k=0, 1, 2\ldots n-1$, on trouve

$$\left|\frac{\partial v_n}{\partial x}\right|<4m\frac{M^n}{(n-1)!}\int_R^P\sum_0^\infty\frac{(x+y)^{n+k-1}}{k!}M^{k+1}\,dy$$

$$<4m\frac{M^{n+1}(x+y)^n}{n!}e^{M(x+y)}<\frac{M^{n+1}(x+y)^n}{n!}.$$

De même pour $\left|\frac{\partial v_n}{\partial y}\right|$.

Les inégalités sont valables pour $k=0$, donc pour toutes les valeurs entières positives de k.

Il s'ensuit comme au § 3, que le procédé des approximations successives est convergent et conduit à la solution du problème.

On le démontrera de la même manière, quand

$$4Me^{m(x+y)}\leqq 1 \quad \ldots\ldots\ldots \tag{20}$$

Cherchons maintenant des conditions, sous lesquelles on pourra affirmer

$$\left|\frac{\partial v_k}{\partial x}\right|<M^{k+1}(x+y)^k, \quad \left|\frac{\partial v_k}{\partial y}\right|<M^{k+1}(x+y)^k,$$

inégalités, qui, quand

$$M(x+y)<1,$$

permettront la même conclusion.

Si ces inégalités valent pour $k=0, 1, \ldots n-1$, on aura

$$\left|\frac{\partial v_n}{\partial x}\right|<4mM^n\int_R^P\sum_0^\infty (x+y)^{n+k-1}M^{k+1}\,dy$$

$$<4mM^{n+1}(x+y)^n\sum_1^\infty\frac{(x+y)^k M^k}{n+k}<4\,m\,M^{n+1}(x+y)^n\,lg\frac{1}{1-M(x+y)},$$

on pourra conclure

$$\left|\frac{\partial v_n}{\partial x}\right|<M^{n+1}(x+y)^n,$$

quand

$$4\,m\,lg\frac{1}{1-M(x+y)}<1\,.\quad\ldots\ldots\ldots\quad(21)$$

De même pour $\left|\frac{\partial v_n}{\partial y}\right|$.

La condition (21) peut s'écrire

$$M(x+y)\leqq 1-e^{-\frac{1}{4m}},$$

donc si elle est satisfaite, on a aussi $M(x+y)<1$.

D'une manière tout semblable on démontrera, que la méthode des approximations successives conduit à la solution, quand

$$m(x+y)\leqq 1-e^{-\frac{1}{4M}}.$$

On pourra traiter de la même manière l'équation

$$\frac{\partial^2 z}{\partial x\,\partial y}=F\left(z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial x}\right),$$

où F représente un polynôme quelconque du deuxième degré à coëfficients, qui sont des fonctions continues de x et y.

Geology. — *Some correcting notes on "Tertiary rocks from the Misool-Archipelago (Dutch East Indies)".* By H. BAGGELAAR. (Communicated by Prof. L. RUTTEN.)

(Communicated at the meeting of February 26, 1938.)

Last year I published a paper in this periodical on tertiary rocks from the Misool-Archipelago (Proc. Royal Neth. Acad. Amsterdam, **40**, 285 (1937)). The occurrence of Orbitoids with lozenge-shaped median chambers in those rocks led me to conclude that the genus *Lepidocyclina* was represented and I further determined some sections as *Spiroclypeus*, though they all showed a very fine structure. Certain suggestions received verbally caused me to have additional thin sections prepared in order to obtain, if possible, greater certainty. Now it seems indeed that there occur foraminifera with both rectangular and spatulate median chambers, in every respect similar to the genus *Orthocyclina* v. D. VLERK, which is, according to VAUGHAN, identical with the subgenus *Asterocyclina*. Some of the new sections convinced me that also the genus *Discocyclina* is present.

Summarizing after examination of the new thin sections and revision of the old ones, I feel obliged to make the following corrections on my first statement.

Balbi leget. Nr. 51 *Discocyclina* sp.

Jef Pelee. Nr. 53*a* *Discocyclina* sp.

Femin (Lola). Nr. 54 *Discocyclina* sp.

Warakarakat. Nrs. 56*b a* and *c* *Discocyclina* sp. and *Asterocyclina* sp.

Kanim. Nr. 57 *Discocyclina* sp.

Sabenibnoe E.-island. Nrs. 58*a a/c* *Discocyclina* sp.

Sabenibnoe W.-island. Nrs. 58*b a, c, e* and *g* *Discocyclina* sp.; Nrs. 58*b b, f* and *i* *Discocyclina* sp. and *Asterocyclina* sp.; Nrs. 58*b d* and *h* *Discocyclina* sp. and a Camerinid, which formerly has been mentioned as *Operculina*, but may perhaps be *Pellatispirella*.

Summary. In so far as the re-examination has yielded a positive result it seems that all rocks are of an *eocene* age.

Geology. — *Geochemistry and the total amount of sediments.* By PH. H. KUENEN. (Communicated by Prof. L. RUTTEN).

(Communicated at the meeting of February 26, 1938.)

In a former paper I tried to calculate the total amount of sediments from the rate of recent sedimentation, as measured by SCHOTT. The amount found was over 20×10^8 km^3. CLARKE and others, and later GOLDSCHMIDT, followed a different method. They calculated the percentage-loss of sodium that igneous rocks sustain on transformation to sediments. Supposing the sodium of the oceans to represent this loss, they found a total amount of sediments of the order of only 3×10^8 km^3.

An attempt should be made to bring these results in closer agreement. Probably the recent sedimentation is somewhat above the average, especially for the tropical Atlantic, where the measurements by SCHOTT were made.

On the other hand various corrections are also needed on CLARKE's result. The sodium content of deepsea deposits is higher than that of continental sediments, but only the latter were used in the sodium method. Most sediments contain salt water in the pore space. Extrusive rocks play a relatively important part in the production of sediments in consequence of their exposed position and loose composition. There are, moreover, a number of considerations that render the result of the sodium method less trustworthy than would appear at first sight.

Weighing the various arguments and the reliability of the methods against eachother, I believe that the total amount of weathered sediments may be estimated at about 8×10^8 km^3. To this amount must be added all sediments formed by mechanical disintegration. Especially before plants successfully cloaked the continents, the importance of disintegration must have been considerably greater than at the present time. A much larger percentage of fine unweathered products was carried by dust storms into the oceans and swelled the bulk of deposits, without adding to the store of sodium of the ocean waters. This would bring the grand total to over 10×10^8 km^3.

The geochemistry of calcium is of special importance. Before the Cambrian the CaO liberated through weathering must have been precipitated in the sea. But from then onwards animal life withdrew the calcium from the oceans and deposited most of it on the continents in the form of shallow water limestones. A considerable store of calcium was thus preserved and a steady and ever increasing circulation was kept going through dissolving by rain water and precipitation by orga-

nisms. During the Cretaceous planktonic Foraminifera first started their activity. Much of the lime was afterwards abstracted from the circulation and permanently buried in the deepsea in the form of Globigerina ooze. The store of lime is thus being gradually used up and calculation shows that in some 100 millions of years a lime famine must set in.

In conclusion the rate of oceanic sedimentation for the geological past can be roughly indicated. Before the Silurian: 1 cm solid in 5000 years, later 1 cm in 10.000 years. Globigerina ooze before the Tertiary negligeable, since then 1 cm in 5000 years.

In a later more detailed paper the author hopes to show how the results here indicated were arrived at, and along which lines further investigations may aid in finding more precise data.

LITERATURE.

PH. H. KUENEN: On the total amount of sedimentation in the deepsea. Am. Journ. Sc., 457—468 (1937).

F. W. CLARKE: The data of geochemistry. Bull. U. Geol. Surv., 770 (1924).

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Medicine. — *Improvement of the method of obtaining neurotoxins out of human urine.* By F. J. NIEUWENHUYZEN. (Experiments from the Institute for Experimental Pathophysiology of the Nervous System. Director: Dr. H. DE JONG, Wilhelmina Gasthuis, Amsterdam) (clinic Prof. Dr. B. BROUWER). (Communicated by Prof. B. BROUWER.)

(Communicated at the meeting of January 29, 1938.)

BOUCHARD (1886) [1]) found that the urine of epileptic patients usually contains a relatively higher toxicity than the urine of normal persons. The toxicity decreases shortly before an attack and is considerably increased after the attack.

SAVERÉ (1907) [2]) found an increased toxicity of the colloidal fraction in the urine of eclamptics, accompanied by an increase in the quantity of colloids.

LÖWE (1911) [3]) determined the excretion of colloids in the urine of normal persons and patients on a large scale. He constantly found an increase of the quantity of non-dialysable substance in the urine of epileptics; sometimes an enormous quantity was present. Moreover, an increased amount of non-dialysable compounds was found in five cases of catatonia and a few cases of hebephrenia, delirium tremens and progressive paralysis. Subcutaneous or intravenous injections into rabbits of small quantities of the colloidal fraction, obtained from catatonic, hebephrenic and paranoid patients, showed a decrease in strength within 2 to 24 hours after the injection. In several cases this decrease in strength is associated with the occurrence of tremors and salivation (autonomic symptoms).

Throughout there was a distinct difference in toxic action between the colloidal fraction out of the urine of catatonic and of epileptic patients. The former never produced epileptic symptoms in experimental animals.

GAMPER and KRALL (1934) [4]) examined the difference in toxicity of non-treated urine from normal and mentally diseared persons in mice. The urine from schizophrenics, persons suffering from manic-depressive psychosis and those with organic lesions proved to have a greater toxicity than the urine of normal persons. The mortality of mice, injected with urine from the first three groups was 43—47 %; the mortality of the mice injected with

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[1]) CH. BOUCHARD, Compt. rend., **102**, 669 (1886).

[2]) M. SAVARÉ, Beitr. z. chem. Physiol. u. Pathol., **9**, 401 (1907).

[3]) S. LÖWE, Z. f. d. ges. Neur. u. Psych., **7**, 73 (1911).

[4]) E. GAMPER and A. KRALL, Z. f. d. ges. Neur. u. Psych., **150**, 252 (1934).

normal human urine was 30 %. Among the different kinds of schizophrenics, the urine of paranoid patients proved to be more toxic than the urine of hebephrenics and catatonic patients.

E. DINGEMANSE, J. FREUD and H. DE JONG (1933—1934) [1]) extracted a large number of alkalinised urines of normal and mentally diseased persons with benzene. They obtained an extract which, when injected into mice and rats, could produce symptoms of experimental catatonia (H. DE JONG and H. BARUK [2]). The smallest quantity of urine, of which the end-product dissolved in olive-oil or water could produce the described symptoms, was defined as a unit.

This examination proved, that the urine of normal individuals contained a greater quantity of toxic products than were present in cases of mentally diseased persons as sufferers of dementia-praecox and manic-depressive psychosis.

In a separate examination of the benzolic extract, obtained from normal individuals, FREUD and DINGEMANSE (1933) [3]) showed that the extracted neurotoxic substance was identical with nicotine. The proof of purity of the isolated substance was determined by quantitative analysis of the nitrogen and finding of the melting-point of a mixture of nicotine-picrate and the picrate of the isolated compound. In the urine of non-smoking normal individuals the mentioned toxic substance was not found. Moreover, by subcutaneous injections with nicotine, the complete syndrome of experimental catatonia could be produced in mice (NIEUWENHUYZEN) (1934) [4]).

TINEL and MARCEL ECK (1933) [5]) could prove the presence of an extremely oxidable and thermolabile toxin in the residue of the benzolic extract of acidified urines of smokers and non-smokers. Subcutaneous injection of an extract, obtained from half a litre of morning-urine, caused diminished motor activity ("stupor") in guinea-pigs and rats, often associated with a true paresis or out-stretched limbs in a tonic state. During this intoxication, which remains for several hours, the cavia does not move but keeps the position in which it is put; remains lying on its back or keeps its balance while standing on the edge of the table. Afterwards often a paresis of the hind limbs occurs with a certain rigidity, while also clonic contractions can follow in the forelimbs. Heart or breathing disturbances were not noticed.

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[1]) J. FREUD and E. DINGEMANSE, Bioch. Z., **255**, 464 (1932); E. DINGEMANSE, J. FREUD and H. DE JONG, Z. f. d. ges. Neur. u. Psych., **143**, 459 (1933); H. DE JONG, Acta Brevia Neerl., **3**, 148 (1933); H. DE JONG, Ned. Tdschr. Gen., **78**, 716 (1934); H. DE JONG, Psych. Neur. Bld. (1934).

[2]) H. DE JONG and H. BARUK, "La catatonie expérimentale par la bulbocapnine". Masson et Cie. Paris (1930).

[3]) E. DINGEMANSE and J. FREUD, Acta Brevia Neerl., **3**, 49 (1933).

[4]) F. J. NIEUWENHUYZEN, Proc. Royal Netherlands Acad. Amsterdam, **37**, 575 (1934).

[5]) J. TINEL, MARCEL ECK and Mrs. ECK, Soc. Méd. Psychol. Paris (1933); Ann. Méd. Psychol., **91**, (II), 710 (1933).

Tinel found this toxin both in the urine of normal persons and of patients. According to his method, he also extracted several normal urine compounds, e.g. phenols, which he did not remove by further purification of the extract. Thus we had to find a method in which normal urine compounds do not pass into the benzolic extract, or could easily be removed.

Therefore we used again the method of Dingemanse et al, which we have thoroughly revised. The non-revised method of Dingemanse et al is briefly as follows:

2—10 litres of urine are made alkaline with a 30 % KOH-solution and extracted three times 3 hours on a boiling water-bath with a backward flowing cooling-apparatus with benzene. The combined benzolic extracts were evaporated until the volume was reduced to less than 200 ml. This was transferred to a separating funnel and shaken out three times with 0.001 normal hydrochloric acid; with this the toxic substance goes into the HCl-solution.

Next the HCl-solution is alkalinised again and shaken out three times with benzene. The benzene is evaporated in vacuum to a small volume and to this so much olive-oil is added, that 1 ml of olive-oil corresponds with an extract of 1 litre urine. After this the mixture oil-benzene is freed entirely from the last benzene-remains on a boiling water-bath and is thus ready for determination of the toxic unit.

EXPERIMENTAL PART.

In the first place it was necessary to prove the exactness of the method, as used by Dingemanse, Freud and de Jong. We have tried to trace and eliminate every source of mistake, which this method could offer. Consequently were examined:

1. The influence of the different methods of conservation upon the toxicity of the urine.
2. The influence of the method of extraction upon the toxicity.
3. The change of toxicity of the extract during the purification.
4. The influence of the solvent, in which the expected toxic substance is solved, on the physiological behaviour of the experimental animals.
5. The influence of the mecanical effect of the injection on the behaviour of the experimental animals ("Injectionshock").
6. The difference in the amount of neurotoxin in the urines from different patients ("Biological testing of the toxin").

1. *Method of conservation of the urine.* The urines were collected in tins or closed jars of 10 litres, so that absorption of nicotine from the air was excluded. As anti-fouling at first thymol was added to the urine, but after experimenting for several months this proved to be insufficient (Miss E. A. Rietmeyer). The thymol added to the urine proved to pass into the benzene-phase when extracted and was not removed by purification of the extract. The result was that the extract, after injection into the experimental animals, produced symptoms of a thymol poisining. See protocol 1.

Protocol 1. Direct injection of thymol.

3 hrs. 00 P.M.: Subcutaneous injection of 1 ml. of saturated thymol solution (0.06 %) into a normal mouse.

3 hrs. 01 The mouse is cataleptic. One can hang the animal by the toes of one hind limb on a horizontal twine. The mouse does not jump away as a normal animal does.

3 hrs. 04 The mouse shows a waggle in walking ("Ataxy"). When the animal is put on the top of a vertical stand, it is not able to hold on to this, but glides slowly downwards (Paresis).

Protocol 2. Thymol passing into the purified benzolic extract.

The benzolic extract is prepared by suspending 10 grams of thymol in 5 litres of water, which was made alkaline with NaOH and extracted with benzene by boiling three times 3 hours. The benzolic extract is transferred to a separating funnel and shaken out three times with 0.1 normal hydrochloric acid. Next the HCl-solution is made alkaline again and shaken out three times with pure benzene. After evaporation of the benzene, the residue is dissolved in 0.01 normal HCl-solution and injected subcutaneously into a white mouse.

1 hrs. 10 P.M.: Injection of the above-mentioned extract. No motor symptoms.

1 hrs. 15 The mouse is negativistic; one can push the animal forward like a block without its walking further. Besides, the mouse is cataleptic; it can be hung up by the fore legs on a horizontally fixed twine without the animal is trying to pull itself against this, as a normal mouse does. Moreover, the mouse gnaws its teeth.

2 hrs. 35 The mouse makes vain attempts to pull itself up, finally it lets itself fall. Besides, the animal has a head tremor and is negativistic.

2 hrs. 50 On touching, the animal has a tremor of the whole body.

3 hrs. 25 Catalepsy and negativism, idem 1 hr. 15. The mouse glides slowly downwards from the stand, is slightly convulsive. Gnawing of the teeth is heard.

Yet, thymol was also shown chemically to be present in the benzolic extract. After strong evaporation of a large quantity of benzolic-extract, crystals were separated while standing in the air, which could easily be shown to be thymol.

Because of this, in future toluene was used. This prevents decay in the urine and can easily be distilled out when the benzolic-extract is evaporated.

2. *Method of extraction of the urine.* After we left off using thymol as anti-fouling of the urine, we tried to find another extraction liquid, instead of benzene. Finally we interchanged ether for the benzene but, because with this also some normal urine-compounds appeared to be extracted, which could not be removed by purifying, we now again use benzene. The commercial benzene was purified from thiophene by distilling with paradimethylaminobenzaldehyde and phosphoric acid.

The urine collected in the way described above, was extracted in a sodium carbonate alkaline medium three times 3 hours in 5 litre jars with a backward flowing cooling-apparatus on a boiling waterbath with benzene. After every extraction care was taken that the urine was still alkaline (blue to thymolphtaleine). If this was not the case, then sodiumcarbonate was added once more, before the next extraction with benzene was done.

The benzolic extract obtained by extracting the urine three times 3 hours, the so-called *"rough benzolic extract"*, contains next to the specific substances several normal urine-compounds, such as phenols and indolderivatives, which have to be separated from the expected toxin. For this the benzolic extract was evaporated until the volume was reduced to less than 200 ml. This was transferred to a separating funnel and shaken out three times with 0.1 normal hydrochloric acid. The toxic substance passes off in the acid watery phase, the indol- and phenol-derivatives stay in the benzene-phase.

Then the watery phase is made alkaline once more (blue to thymol-phtaleine) and shaken out three times with thiophene-free benzene. The combined benzene-phase is washed out with distilled water till the last trace of alkaline is removed and brought over in a tall measuring jar. After the last water drops have fully sunken down out of the benzene, the benzene-phase is poured off into a flask containing $2\frac{1}{2}$ ml. of olive-oil and the benzene was distilled off.

The olive-oil must, immediately after the evaporation of the benzene be injected into the experimental animals, because with cooling down below 37° C, part of the dissolved substance may precipitate out of the olive-oil.

3. *Change of the toxicity of the extract during purification.* The influence of alkalis and acids proved to us to be of great importance in the process of preparing and purification of the benzolic extract. So the benzolic extract may not be long in contact with diluted HCl, as we have noticed that through this, chemical changes happened in the benzene-content. This can be shown in the following way:

Protocol 3. A fresh benzolic extract, shortly shaken out with 0.1 normal hydrochloric acid, gives an intense red colour with EHRLICH's reagent. A benzolic extract, that has been in contact with 0.1 normal hydrochloric acid for several hours, does not give this colour reaction.

Sensibility to alkali. The degree of alkalinity, in which we have extracted the urine with benzene, proved to be of very great importance on the toxicity of the purified extract.

To prove the influence of alkalis on the amount of toxin of the benzolic extract, several samples of the same urine series, which were obtained from one patient, were made alkaline with a 30 % KOH-solution or with a sodium-carbonate solution and extracted separately with benzene. The result of this examination was that in case of the mentioned patient there

proved to be regularly present a strongly reacting neurotoxin in the benzolic urine-extract, if only extracted in sodium-carbonate alkaline solution. When the reaction was strongly alkaline, no neurotoxin could be proved to be present in the purified benzolic extract.

Protocol 4. The benzolic extract of 6 litres of urine of patient W. (liver-cirrhosis), prepared in sodium-carbonate alkaline solution, is shaken out three times with 0.1 normal hydrochloric acid. After this, the acid watery phase is again alkalinised with sodium-carbonate (blue to thymolphtaleine) and shaken out three times with pure benzene. The combined benzolic extract is washed out with distilled water till the last trace of alkali is removed. After this the benzene content is dissolved in 3 ml. of olive-oil by evaporation from the benzene on a boiling water bath and injected subcutaneously into white mice.

I. Normal mouse with bodyweight of 15 grams.

2 hrs. 35 P.M.: Injection of 1 litre equivalent extract. The mouse behaved quite normally after the injection.
3 hrs. 00 *Mouse is strongly paretic.*
3 hrs. 15 Mouse showed a mixture of *paresis* and *negativism*: the animal walks backwards after being pushed forward.

The next day: *the mouse died.*

II. Normal mouse, bodyweight of 15 grams.

2 hrs. 40 P.M.: Injection of 2 litre equivalent extract. The mouse behaved altogether normally after the injection.
3 hrs. 05 *The mouse is strongly paretic.*
3 hrs. 10 *The mouse develops a slight tremor.*
3 hrs. 20 Strong tremor on being touched. If put on its back, the animal is not able to turn itself over.

Next day: *the mouse died.*

III. Normal mouse, bodyweight of 15 grams.

2 hrs. 45 P.M.: Injection of 2 litre equivalent extract. The mouse behaved altogether normally after the injection.
3 hrs. 10 The mouse still behaves normally.
3 hrs. 20 *The mouse is strongly paretic.*
3 hrs. 30 *Convulsions set in, followed by death.*

Conclusion: After subcutaneous injection of a benzolic extract of 1 litre of urine from patient W., prepared in a sodium-carbonate alkaline solution and purified, pronounced signs of intoxication in the form of negativism and paresis occurred. After injection of a greater quantity even death followed.

Protocol 5. After this the same quantity of urine (6 litres) from patient W. was made alkaline with a 30 % KOH-solution (blue to thymolphtaleine), extracted in the same prepared way and purified further as with the benzolic extract in sodium-carbonate alkaline solution. The purified benzene content is dissolved in 3 ml. of olive-oil and injected subcutaneously into white mice.

I. Normal mouse, bodyweight of 15 grams.

3 hrs. 50 P.M.: Injection of 2 litres equivalent extract. The mouse behaves absolutely normally after the injection.
4 hrs. 15 The mouse behaves still absolutely normally.
4 hrs. 45 Idem.
5 hrs. 15 Idem.

II. Normal mouse, bodyweight of 15 grams.

4 hrs. 00 P.M.: Injection of 2 litre equivalent extract. The mouse behaves absolutely normally after the injection.
4 hrs. 15 The mouse behaves still absolutely normally.
4 hrs. 45 Idem.
5 hrs. 15 Idem.

III. Normal mouse, bodyweight of 15 grams.

4 hrs. 03 P.M.: Injection of 1.6 litre equivalent extract. The mouse behaves altogether normally after the injection.
4 hrs. 15 The mouse behaves still absolutely normally.
4 hrs. 45 Idem.
5 hrs. 15 Idem.

Conclusion: Urine which, when extracted in sodium-carbonate alkaline solution with benzene, gives a strong neurotoxic extract in the animal experiment, gives a toxin-free extract when prepared in a stronger alkaline-solution. Probably does a higher degree of alkalinity destroys the toxin.

4. *Influence of the solvent, in which the expected toxic substance is dissolved on the physiological behaviour of the experimental animals.* The most suited injection-solution for experimental animals proved to us to be olive-oil. Controling tests taught us that olive-oil is an indifferent injection substance for mice. When olive-oil is mixed with pure benzene and the benzene is distilled off, the olive-oil does not contain a toxic substance. By mixing olive-oil with 200 ml. of thiophene-free benzene, followed by evaporation of the benzene on a boiling waterbath and subcutaneous injection of the oil-residue into white mice, no other result was obtained except the effect of the mechanical prick ("Injection-shock"). After pricking mice in the back or in the flank, it may for instance happen that the animals remain hanging with stretched backs, when hung by the front legs on a horizontally stretched piece of twine. On the contrary, a normal non-injected mouse pulls itself up immediately, so that it fixes itself with the four feet on the twine.

Protocol 6. 200 ml. of thiophene-free benzene is evaporated on a boiling waterbath with 3 ml. of olive-oil. 1 ml. of this is injected subcutaneously into 3 white mice.

I. Normal mouse. The animal climbs normally up a twine before the injection.

1 hr. 30 P.M.: Injection of 1 ml of olive-oil, treated in the above-mentioned way. After the injection, the mouse normally climbs up against the twine.

2 hrs. 00 The mouse behaves still absolutely normally.
2 hrs. 30 Idem.
3 hrs. 30 Idem.

II. Normal mouse. The animal climbs up normally against a twine before the injection.

1 hr. 30 P.M.: Injection of 1 ml. of olive-oil, treated in the above-mentioned way. After the injection the mouse remains hanging in stretched attitude on the horizontal twine. The mouse shows no symptoms of poisining.
2 hrs. 00 The mouse behaves in the same way as 1 hr. 30.
2 hrs. 30 Idem.
3 hrs. 00 Idem.

III. Normal mouse. The mouse climbs up normally against the twine before the injection.

1 hr. 30 P.M.: Injection of 1 ml. of olive-oil, treated in the above-mentioned way. After the injection the mouse remains hanging in stretched attitude on the horizontal twine. The mouse shows no signs of poisining.
2 hrs. 00 The mouse behaves in the same way as 1 hr. 30.
2 hrs. 30 Idem.
3 hrs. 00 Idem.

Conclusion: Mixing olive-oil with pure benzene, followed by evaporation of the benzene and subcutaneous injection of the oil-residue into white mice, produced no other symptoms than the mechanical effect of the prick. See protocol 7.

5. *Influence of the mechanical effect of the injection on the behaviour of the experimental animals ("Injectionshock").* After pricking mice in the back or in the flank, it may sometimes happen that the animals remain hanging with stretched backs, when they are hung up by the front legs on a horizontally stretched piece of twine. On the contrary, a normal non-injected mouse pulls itself up immediately, so that it fixes itself with the four feet on the twine. See the following protocol:

Protocol 7.

I. Normal mouse. The animal climbs normally up a twine before the injection.

10 hrs. 52 A.M.: Prick with an injection needle into a fold of the skin of the back. After the prick the mouse remains hanging in a stretched attitude on the horizontal line. No other pathological symptoms were found.

II. Normal mouse. The animal climbs up normally against a twine before the injection.

10 hrs. 55 A.M.: Prick with an injection needle into a skin fold of the flank. After the prick the mouse remains hanging in an extended attitude on the horizontal twine. Nothing else abnormal to be found.

11 hrs. 00 The mouse again climbed up normally against the horizontal twine, so that it fixed itself with its four feet on the twine.

12 hrs. 00 The mouse climbed up against the twine, idem 11 hrs. 05.

III. Normal mouse. The animal climbs up normally against a twine before the injection.

11 hrs. 12 A.M.: A prick with an injection needle into a skin fold of the back skin. The mouse also pulls itself up normally against the twine after the prick.

11 hrs. 20 Once more the mouse receives a prick under the skin of the back, combined with the inflation of 2 ml. of air. After the injection the mouse still pulled itself up normally against the horizontal twine.

6, *Biological standardization of neurotoxins in urine.* For the determination of the quantitiy of toxin extractable with benzene and the quality of the toxic reaction, we made use of the animal experiment. To be able to use the smallest possible quantity of urine, we preferred to use white mice of 15 grams bodyweight as experimental animals.

It is of great importance that one does not keep the animals too long in hand. H. DE JONG described in 1933 [1]) how a "hypnotic catalepsy" in rats, mice and rabbits could be mistaken for a catalepsy brought about by chemical substances. Herefore the mice were examined before and after the injection to see if they showed a "hypnotic catalepsy". This happened by turning the mice about three times quickly on their sides. A normal mouse must lift itself up again immediately when one lets it go. If the mice did not do this, they were not used for the standardization test of the urine extracts.

As a criterium for the existence of *catalepsy*, we have taken "the holding itself up to a vertical stand. When one puts a normal mouse with the head downwards against the top of a vertical stand, it walks downwards, either spontaneously or after a single prick in its tail. A cataleptic mouse remains sitting, clinging to the stand and does not even walk down after several pinches in its tail. A paretic mouse holds itself on to the stand, it glides downwards along the iron.

The behaviour of the mice on the stand seems to us to be the best test for catalepsy, as a cataleptic attitude against the stand could never be obtained separately by a mechanical prick.

In opposition with this, however, remaining in the same position, when hanging on a horizontal twine, as the only symptom, may not be considered to be evidence for catalepsy, because a simular attitude can also be obtained by mechanical pricking.

In our experimental series it was evident that not lifting itself up, when the mouse is put down lying on its side, is always accompanied by *paresis*

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[1]) H. DE JONG, Psych. Neur. Bld. (1934).

in the hind legs, so that the mouse cannot fix itself with the hind legs on a vertical stand or a horizontal twine.

We considered the animals as *"negativistic"*, when they showed an active resistance to every change of position. This becomes apparent when one can push them forwards like a block and the animals resist change of attitude.

For every biological standardization three mice were injected with so much extract as corresponds with $\frac{1}{2}$ to 2 litres of urine.

We carefully noticed that the urine we examined were from non-smokers and also from persons taking no medicine.

A picture of the biological standardization of benzolic extracts of a number of urines, prepared with strong alkaline and sodium-carbonate alkaline solutions, may be found in the columns I and II. Next to the

COLUMN I.

Biological testing of neurotoxin soluble in benzene from the urine, injected into mice.

The neurotoxin is extracted by strong alkaline reaction of the urine. Next to the smallest quantity of toxic urine, in figures the number of mice is given, that shows the symptoms described below.

Nr.	Diagnosis:	Minimum dose of toxic urine	Clinical symptoms in 3 mice.					
			Catalepsy	Negativism	Parasis	Tremor	Convulsions	Exit
54. W	Psychosis	—	0	0	0	0	0	0
55. E	Cataton. Schiz.	—	0	0	0	0	0	0
56. W	The same as 54	—	0	0	0	0	0	0
59. V	Schizophrenia	2 L	0	0	1	0	1	0
60. D	Schizophrenia	1 L	0	0	0	1	2	2
62. G	Psychosis	—	0	0	0	0	0	0
63. K	Schizophrenia	—	0	0	0	0	0	0
64. H	Schizophrenia	—	0	0	0	0	0	0
65. B	Schizophrenia	0.2 L	1	0	0	0	0	0
67. W	Schizophrenia	1 L	0	0	0	1	0	0
68. B	Schizophrenia	—	0	0	0	0	0	0
69. G	Schizophrenia	1 L	0	1	0	0	0	0
70. B	The same as 68	—	0	0	0	0	0	0
71. G	The same as 69	2 L	0	1	0	0	0	0
72.	Healthy	—	0	0	0	0	0	0
73. X	Schizophrenia	—	0	0	0	0	0	0
76. K	Schizophrenia	2 L	0	0	1	2	1	2

COLUMN II.

Biological testing of neurotoxin soluble in benzene from the urine, injected into mice.

The neurotoxin is extracted by sodium carbonate alkaline reaction of the urine. Next to the smallest quantity toxic reacting urine, in figures the number of mice is given, that shows the symptoms described below.

Nr.	Diagnosis:	Minimum dose of toxic urine	Clinial symptoms in 3 mice.					
			Cata lepsy	Negati-vism	Paresis	Tremor	Convul-sions	Exit
78. K	Schizophrenia	—	0	0	0	0	0	0
80. B	Schizophrenia	—	0	0	0	0	0	0
81. K	Schizophrenia	0.8 L	1	1	2	1	2	0
82. B	Schizophrenia	0.6 L	0	2	0	2	2	1
83. B	Schiz. Epileps.	—	0	0	0	0	0	0
84. B	Schizophrenia	2 L	1	1	0	0	0	0
85. V	Cataton. Schiz.	0.6 L	1	0	2	1	0	0
86. D	Hebephr. Schiz.	1 L	0	0	0	1	1	1
87. L	Cataton. Schiz.	0.8 L	0	0	0	3	0	0
88. V	Cataton. Schiz.	0.5 L	3	0	2	0	1	0
89. V	Schizophrenia	1 L	0	1	2	1	0	0
90. H	Cataton. Schiz.	0.8 L	1	0	1	2	0	0
92. G	Dem. paranoid.	1 L	0	0	1	1	0	0
93. H	Schizophrenia	—	0	0	0	0	0	0
94. H	Cataton. Schiz.	1 L	0	2	1	1	0	0
96. H	94. after 3 months	—	0	0	0	0	0	0
95. O	Schizophrenia	—	0	0	0	0	0	0
97. B	Schizophrenia	—	0	0	0	0	0	0
98. H	Hebephr. Schiz.	—	0	0	0	0	0	0
99. G	Schizophrenia	—	0	0	0	0	0	0
100. B	Schizophrenia	0.5 L	2	0	0	2	0	0
101. V	Schizophrenia	0.3 L	0	0	0	0	0	3
102. H	Cataton. Schiz.	0.4 L	0	0	0	0	0	2
103. J	Hebephr. Schiz.	—	0	0	0	0	0	0

smallest quantity of toxic urine, the number of mice is given which showed the clinical symptoms mentioned.

A comparison of tables I and II proved, that, in strong alkaline solution,

in only 41 % of the 17 examined cases a neurotoxin could be extracted out of the urine by benzene. By sodium carbonate alkaline extraction with benzene the amount of toxin of the purified extract was greater, so that here in 58 % of the 24 examined cases a neurotoxin could be shown to be present in the urine extracts.

Resume.

According to the method described above we mean to have improved the methods to obtain neurotoxins out of human urine, by doing away with the different mistakes in the methods wich we have discovered in the course of time in our laboratory.

The application of this method together with the use of clinical material will be published later.

Medicine. — *Neurotoxic symptoms, especially catatonia, produced in mice by substances out of the human urine.* By F. J. NIEUWENHUYZEN. From the Institute for Experimental Pathophysiology of the Nervous System (Dr. H. DE JONG) of the Neurological Clinic of the University of Amsterdam (Prof. Dr. B. BROUWER). (Communicated by Prof. B. BROUWER).

(Communicated at the meeting of January 29, 1938.)

The examinations in our Institute lead us of recent years to study the neurotoxic action of urine extracts of normal persons and patients. Especially of a large number of Dementia-praecox patients the neurotoxic action of urine extracts was studied in experimental animals. The question whether the excretion of a specific neurotoxic substance is increased in this disease, can be examined in several ways:

1. By extracting the toxin out of the urine with a liquid, which is not soluble in water, evaporating the solvent and purifying the residue by recrystallization, to be continued till the toxic action in the animal experiment does no longer increase. After this the purified substance is to be subjected to a chemical analysis.

2. A second method is the examination of all obtainable normal and pathological urine derivatives of known chemical composition in experimental animals on their neurotoxic action till the compound is found that gives the same syndrome of intoxication in the animal experiment with the same dosage as the substance isolated out of the urine.

The result of tests, concerning the first series of substances examined in this connection, follows below.

Urea. Although several cases are known, in which catatonic symptoms were observed in uraemic patients, we by no means considered urea to be responsible for the catatonic symptoms.

Urea itself, proved to be relatively harmless after subcutaneous injection into mice, which agrees with the general experience.

It was only after subcutaneous injection of 10 grams of urea per kg bodyweight, i.e. 150 mgr of urea into a mouse of 15 gr, that catatonic symptoms were seen. The column below gives an impression of the motor symptoms of poisining in mice:

Nov. 13th, 1933.

12 hrs. 56 P.M.: Injection of 160 mgr of urea, dissolved in 0.8 cc of aqua dest. into a mouse of 18 gr bodyweight.

12 hrs. 58 — The mouse is cataleptic. When the mouse is placed on top of a high vertical stand, it does not walk downwards to the bottom as before the injection, but remains sitting quietly, even if it is pinched in its tail. If the mouse is turned sidewards, lying so that one of the hind limbs sticks up in the air, this is not drawn backwards under the body after the animal is allowed to go.
1 hr. 00 — Cat alepsy the same as at 12 hrs. 58.
1 hr. 02 — Motor iniative returned: the mouse sits cleaning itself and climbs up spontaneously against the vertical axis of the stand.
1 hr. 45 — Motor iniative totally normal.

Uric acid. Although it was very improbable that uric acid was able to produce motor disturbanes in experimental animals, which could be considered as catatonic symptoms, we injected 1 cc of saturated sodium urate solution subcutaneously into a number of white mice.

July, 22nd, 1937.
Normal mouse with bodyweight of 16 gr.
3 hrs. 45 P.M.: Subcutaneous injection of 1 cc of newly prepared saturated sodium urate solution in Ringers solution.
4 hrs. 00 — Motility perfectly normal.
4 hrs. 30 — Idem.
5 hrs. 00 — Idem.
6 hrs. 00 — Idem.
Conclusion: No motor abnormalities.

Creatinine. As it is known that creatinine plays a very important part in the function of muscles, we studied experimentally the influence of creatinine injections on the motor function of mice and guinea pigs. The result was that we observed no motor symptoms, not even after intravenous injections of a 10 % solution of creatinine in Ringers solution.

Oxyprotein-acid fraction. By examining the neurotoxic qualities of the normal urine substances, our attention was drawn to the hardly chemically known oxyprotein-acid fraction. GULOTTA (1930) [1]) found an increased quantity of antoxyprotein-acid, while determining the quantity of the oxyprotein acids in the urine of two dementia praecox patients. On the ground of this observation it seemed to us to pay to isolate the oxyprotein-acid fraction out of the urine of normal persons and dementia praecox patients and to examine the probable neurotoxic action in experimental animals.

The result was that the oxyprotein acids, obtained out of the urines of several patients, had a very strong neurotoxic action. The quantity of oxyprotein acid obtained out of 1 litre of urine proved to be able to poison 17 mice, so that the toxic unit of the maximally purified preparation was 6 mgr which was necessary for one mouse of 16—18 grams. The symptoms

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[1]) S. GULLOTTA, Arch. f. Psych., **90**, 436 (1930).

of the maximally purified preparation was a definite catalepsy without the occurrence of convulsions.

Technique: The exprotein acids are characterized chemically by the solubility of their barium- and silver salts in water and the insolubility of their barium and silver salts in absolute alcohol. On this the principle of the isolation of the oxyprotein acids out of urine depends.

For further details we point to the method of O. VON FÜRTH [1]), which we have followed. The bariumoxyproteinates obtained by this method were changed by us into sodium salts. The purified sodium oxyproteinates were dissolved in Ringers solution and injected subcutaneously into white mice.

July 21st, 1934.

I. Normal mouse, bodyweight 18 gr.

10 hrs. 30 A.M.:	Subcutaneous injection of a quantity of oxyprotein acid derived from 0.168 litre of urine, dissolved in 1 cc 0.9 % Na Cl.
10 hrs. 40	The mouse is not able to pull itself up a horizontal twine.
10 hrs. 55	*The mouse shows a mixture of catalepsy and hyperkinesis.* One can hang the animal by the toes of the hind legs on a horizontal twine, without its falling down. If placed on top of a vertical stand, the mouse jumps down from it spontaneously (*Hyperkinesis*).
11 hrs. 10	The mouse remains hanging with both forelimbs on a horizontal twine, pulls itself up when pinched in the tail.
11 hrs. 35	Idem 11 hrs. 10′.
12 hrs. 00	Beginning of paresis: when put on top of a vertical stand, the mouse goes down, partly walking, partly gliding downwards.
12 hrs. 50 P.M.:	Idem. 12 hrs. 00.

II. Normal mouse, bodyweight 25 gr.

3 hrs. 42 P.M.:	Subcutaneous injection of a quantity of oxyprotein acid obtained 0.200 litre of urine.
4 hrs. 00	*The mouse is strongly cataleptic.* If one puts the animal in sideward lying attitude, so that one of the hind limbs sticks up in the air, this is no longer pulled backwards under the body after the mouse is allowed to go.
4 hrs. 43	Catalepsy the same as at 16 hrs. 00.

III. Normal mouse, bodyweight 16 gr.

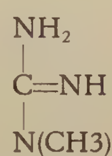
11 hrs. 20 A.M.:	Subcutaneous injection of a quantity of oxyprotein acid obtained from 0.060 litre of urine.
11 hrs. 35	Motility still absolutely normal.
11 hrs. 55	The mouse is *negativistic*: the animal walks backwards after it is pushed forwards. Placed on top of a veryical stand, the mouse slowly glides downwards (Beginning of paresis).
12 hrs. 25 P.M.:	The mouse is not able to pull itself up against a horizontal piece of twine. Placed on the stand, the animal glides downwards, idem 11 hrs. 55′.
2 hrs. 45 P.M.:	The mouse is found dead.

[2]) O. VON FÜRTH, Bioch. Ztschr., 69, 448 (1915).

IV. Normal mouse, bodyweight 15 gr.
11 hrs. 12 A.M.: Subcutaneous injection of a quantity of oxyprotein acid obtained from 0.040 litre of urine.
11 hrs. 30 Motility absolutely normal.
11 hrs. 55 Idem.
12 hrs. 30 P.M.: Idem.
2 hrs. 45 Motility still normal.

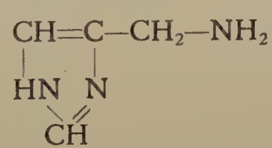
Conclusion: After subcutaneous injection of a quantity of oxyprotein acid out of 0.060 litre of urine, corresponding with 6 mg of the maximally purified preparation, neurotoxic symptoms of negativism and slight paresis were seen in white mice. After injection of a higher dose, catalepsy also occurred and hyperkinesis.

If one compares the found toxicity of the oxyprotein-acid fraction with the toxic dose of the known most toxic substances in the urine, the oxyprotein acid-fraction seems to be on the same line with this. As specific toxic compounds in the urine we examined: dimethylguanidine, histamine, indolethylamine and nicotine.



Dimethylguanidine.

Dimethylguanidine. The symptoms of this have been described previously (E. A. RIETMEYER and F. J. NIEUWENHUYZEN, 1936) [1]. Here is mentioned only that the symptoms after subcutaneous injection of dimethylguanidine chloride into mice consisted of: walking on tiptoe, kangeroo attitudes, hyperkinesis, ataxia, tremors, negativism and autonomic symptoms (salivation). The minimal toxic dose for mice of 15 grams amounted to 2 mgr of dimethylguanidine chloride.



Histamine.

Histamine: The occurence of this in the urine of dementia praecox patients was supposed by BUSCAINO [2]). In order to prove the influence of histamine on the motor actions, a series of mice were injected subcutaneously with a 0.1 % histaminechloride solution.

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[1]) E. A. RIETMEYER and F. J. NIEUWENHUYZEN, Proc. Royal Netherlands Acad. Amsterdam, **39**, 281 (1936).

[2]) V. M. BUSCAINO, Ztschr. f. d. ges. Neur. u. Psych., **125**, 734 (1930).

Dosages of 1.5 mgr of histamine or less had no visible effect on mice of 20 grams bodyweight; with a dosage of 2 mgr negativism and catalepsy occurred, the last was always associated with a beginning of paresis.

December 19th, 1933.

I. Normal mouse, bodyweight 21 gr.

3 hrs. 25 P.M.: Subcutaneous injection of 1.5 cc 0.1 % histamine-hydrochloride solution.

3 hrs. 37 Motility absolutely normal.

4 hrs. 10 Idem.

4 hrs. 30 Idem.

II. Normal mouse, bodyweight 20 gr.

12 hrs. 05 P.M.: Subcutaneous injection of 2 cc 0.1 % histamine-hydrochloride solution.

12 hrs. 10 Motility normal.

12 hrs. 20 Motility still normal.

12 hrs. 30 *The mouse shows mixed symptoms of catalepsy and paresis.* One can hang the mouse up by one hind limb on a horizontal piece of twine; the mouse encircles the thread with his toes and remains hanging in this attitude. If one turns the animal sidewards, so that one of the hind limbs sticks up in the air, this is not pulled backwards under the body after one lets the mouse go. When the mouse is put on top of a vertical stand, it glides downwards, partly walking and partly gliding.

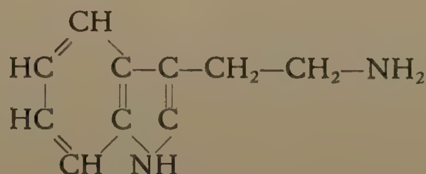
12 hrs. 45 Catalepsy idem 12 hrs. 30.

1 hr. 15 P.M.: Catalepsy idem 12 hrs. 30. *The mouse allows itself to be hung up by one toe of a hind limb* on the horizontal twine; the animal does not move and remains hanging on to the twine. When put on top of a stand, the mouse drops downwards, partly walking and partly gliding.

1 hr. 40 The mouse remains hanging with his left hind leg on the twine and holds with the other hind limb on to the left forelimb. Moreover, the mouse is *negativistic*: the animal resists to be moved forwards and allows itself to be pushed forwards like a block. When placed on top of the stand, the mouse goes downwards, partly walking and partly gliding.

2 hrs. 40 Catalepsy and negativism idem 13 hrs. 40, besides slight paresis.

Conclusion: After subcutaneous injection of 2 mgr of histamine hydrochloride in mice, catatonic symptoms, such as catalepsy, negativism and paresis, occurred.



Indolethylamine.

Indolethylamine. As SULLIVAN (1922) [1]) found indolethylamine in the urine of pellagra-patients and as catatonic symptoms occur in pellagra-patients as well as in Dementia-praecox patients, we examined the probable catatonic action of indolethylamine-hydrochloride experimentally in cats and white mice. The result was that indolethylamine proved to be able to produce the complete syndrome of experimental catatonia in cats and mice. The observation of the experimental indolethylamine catatonia in cats is already published elsewhere (F. J. NIEUWENHUYZEN, 1936) [2]), so that we shall here exclusively consider the results obtained from mice.

January 29th, 1935.

I. Normal mouse.

10 hrs. 20 A.M.: Subcutaneous injection of 2 mgr of indolethylamine-hydrochloride.
10 hrs. 25 Motility absolutely normal.
10 hrs. 35 Idem.
12 hrs. 25 P.M.: Idem.

II. Normal mouse.

10 hrs. 25 A.M.: Subcutaneous injection of 4 mgr of indolethylamine-hydrochloride.
10 hrs. 35 Motility absolutely normal.
12 hrs. 25 P.M.: Idem.

III. Normal mouse.

10 hrs. 32 A.M.: Subcutaneous injection of 6 mgr of indolethylamine-hydrochloride.
12 hrs. 20 P.M.: Motility still normal.
3 hrs. 20 *The mouse shows mixed symptoms of catalepsy and paresis.* If we turn the animal sidewards, so that one of the hind limbs sticks up in the air, this in not pulled back under the body after the mouse is let loose. Placed on top of a vertical stand, the mouse goes downwards, partly walking and partly gliding. If one hangs the mouse by its forelimbs on a horizontal twine, the animal climbs up against this, jumps down and reaches the ground in safety.

IV. Normal mouse.

10 hrs. 41 A.M.: Subcutaneous injection of 10 mgr of indolethylamine-hydrochloride.
10 hrs. 50 Motility still normal.
12 hrs. 20 P.M.: *The mouse shows a mixture of catalepsy and paresis.* If we turn the animal sidewards, so that one of the hind limbs sticks up in the air, this is not pulled backwards under the body after the mouse is let loose. When placed on top of a vertical stand, the mouse goes downwards, partly walking and partly gliding. If we hang the mouse by its forelegs on the horizontal twine, the animal remains hanging without moving.

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[1]) M. X. SULLIVAN, Journ. Biol. Chem., **50**, 39 (1922).

[2]) F. J. NIEUWENHUYZEN, Proc. Royal Netherlands Acad. Amsterdam, **39**, 1151 (1936).

3 hrs. 35	Rigidity due to paresis of the hindpart. The mouse pulls itself forwards with both forelimbs, while the hind limbs as well as the whole hindpart shows signs of stiffness.

Conclusion: By subcutaneous injection of indolaethylamine-hydrochloride into white mice the whole syndrome of experimental catatonia could be obtained. The smallest dose after which motor symptoms were witnessed was 6 mgr of indolaethylamine-hydrochloride. With a higher dosage no hyperkinesis occurred but rigid paresis.

Nicotine: The action of this in mice and rats was examined already several times in succession. For this see the work of FREUD and DINGEMANSE (1933) [1]) and F. J. NIEUWENHUYZEN (1934) [2]). Here is mentioned only that also nicotine proved to be able to produce the complete syndrome of experimental catatonia in mice.

Characteristic of nicotine-catatonia, however, tremors are most outstanding, while the cataleptic phase is not so pronounced. The smallest toxic dose in our experiments with white mice was 0.040 mgr of nicotine-hydrochloride.

November 7th, 1933.
Normal mouse, 20 grams.

10 hrs. 03 A.M.:	Subcutaneous injection of 0.150 mgr of nicotine.
10 hrs. 08	*The mouse is definitely negativistic, yet shows no signs of catalepsy.*
10 hrs. 25	Idem as 10 hrs. 08.
10 hrs. 35	The mouse is negativistic, idem 10 hrs. 08. Moreover, the mouse shows signs of an action tremor.
10 hrs. 48	The mouse shows a strong *actiontremor*, after being pinched in the tail.
10 hrs. 53	*The mouse shows negativism and a strong action-tremor accompanied by catalepsy.* When the mouse is placed on top of a vertical stand, the animal does not walk downwards as a normal mouse does, but remains sitting quietly, even if pinched several times in the tail.
11 hrs. 45	Motor initiative came back again: when placed on top of the stand, the mouse walks down normally. Moreover, the tremors and negativism have disappeared.

Conclusion: By subcutaneous injection of nicotine-hydrochloride into white mice, all symptoms of experimental catatonia could be obtained: negativism, catalepsy and tremors.

Resume.

In the above-described series of experiments is demonstrated that several

[1]) E. DINGEMANSE and J. FREUD, Acta Brevia Neerl., **3**, 49 (1933).

[2]) F. J. NIEUWENHUYZEN, Proc. Royal Netherlands Acad. Amsterdam, **37**, 575 (1934).

normal and pathological urine components of known chemical composition are able to produce neurotoxic symptoms in experimental animals. Moreover, catatonic symptoms occurred; the toxic dose, however, is very different for the various examined substances.

Toxic dose.

Urea	150 mgr		Dimethylguanidine	2 mgr
Creatinine	> 100 mgr		Histamine	2 mgr
Oxyprotein acid	6 mgr		Nicotine	0.04 mgr
Indolethylamine	6 mgr			

Comparative Pathology. — *Dimension and Form with the growth of the seeds of Phaseolus vulgaris.* I. By G. P. FRETS. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of February 26, 1938.)

In the summer of 1937 a number of beans of pure lines I and II [1]) were measured and weighed in their growth and the indices determined. The intention was to check to what extent the dimensions increase independently of each other [2]). JOHANNSEN in his many years' work with pure lines of beans originally placed the dimensions first in his conception of this problem (1913). Later (1926) he spoke of form units [3]).

From the various plants of the cultivation of the I- and of the II-line we picked some growing green pods and measured and weighed the beans. Small and large. The fresh beans are very watery; the dimensions and weights of full-grown fresh beans are considerably larger than those of dried beans. We should have liked the number of small beans that were investigated to have been greater but it is sufficient for the study of our problem. The indices have been calculated. It appears that in judging the results we must take into consideration the spurious correlation [4]).

We have drawn up correlation tables for L and LB (tables 1—3). We allowed the lengths to increase by 0.1 mm, the indices by 0.5 unit. Table 1 contains the length and indices of the first picking of beans of the I-line; upwards of 600 small and large and relatively few medium-size beans. The length varies from 2.9—22.9 mm, the LB-index from 52.5—74.

Table 2 contains the length and indices of the picked beans of the II-line. Of upwards of 650 beans the length varies from 2.9—20.4 mm, the LB-index from 56.5—89.

Table 3 contains the length and indices of a second later picking of beans of the I-line, consisting almost entirely of large beans. Of nearly 500 beans the length varies from 11.3—24.6 mm, the LB-index from 49.5—73.5. A second picking of the II-line was not made as the II-beans at this moment were all quite ripe.

From the correlation tables we can see to what extent the dimensions increase independently of each other. Likewise from the tables which give a review of the average indices belonging to increasing length classes.

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[1]) Genetica, Vol. **16**, 46 (1934).

[2]) D'ARCY W. THOMSON, Growth and Form (1917).

[3]) Elemente der exakten Erblichkeitslehre, 2. und 3. Aufl.

[4]) K. PEARSON, On a form of spurious correlation etc. Proc. Royal Soc., Vol. **60**, 489 (1897).

TABLE 1. Simplified correlation table 1. Beans in growth. 1937. I-line. First picking. The length classes increase by 1 mm, the index classes bij 2.5 units. In the orginal tables 1—3 the length increases by 0 1 mm, the index by 0.5 unit.

L	Ind. 50	52.5	55	57.5	60	62.5	65	67.5	70	72.5	75	Total
30				2	1							3
40	1	1	1	1	2		3					9
50		1	6	6	6	2	2	2				25
60			4	9	6	2	2	2		1		26
70			3	6	12	9	9	8	1	1		49
80				1	4	16	12	13	3	1		50
90			1	2	2	10	10	8	5	1		39
100					1	10	13	13	1	1		39
110				1	3	10	10	9	1			34
120						6	8	9	3			26
130					2	9	8	8				27
140				1	3	10	13	4				31
150				1	12	17	9	3				42
160				1	7	11	8	2				29
170				5	16	17	11	3				52
180				3	12	10	6	5				36
190			2	4	7	15	7					35
200			1	2	7	8	1	1				20
210		1	2	7	8	6	1	1				26
220				1	2	4	1					8
230	1					1						2
Total	2	3	20	53	113	173	134	91	14	5		608

(meth. of JOHANNSEN). From the correlation tables for the successive length classes (I took as class size 1 mm). I have calculated the average indices. (Table 4).

The correlation tables (tables 1 and 2) show something peculiar. Beginning with the somewhat greater lengths and proceeding to the greatest, we see that the indices go down a little from high to low; there is thus a negative correlation. For the short lengths this is not so; there the sequence is rather the reverse and there is thus a positive correlation;

TABLE 2. Simplified correlation table 2. II-line. Beans in their growth 1937. Correlation of L and LB. The length classes increase by 1 mm, the index classes by 2.5 units.

L	Ind. 55	57.5	60	62.5	65	67.5	70	72.5	75	77.5	80	82.5	85	87.5	90	Total
30							4	1								5
40				1	1	3		1	5	2	1					14
50				1	1	3	9	3	1	2	5	3	2		1	31
60							3	1	5	2	6	2	3	1	1	24
70					1		3	2	7	7	8	5	6	1		40
80								2	3	11	11	3				30
90				1					5	11	5	1		1		24
100									11	17	12	1	1			42
110							1	1	7	15	6	1				31
120						2	2	14	23	10	2					53
130					1	1	9	17	16	1	2					47
140						8	17	26	26	7	4					88
150					6	5	21	49	10	8	3					102
160			1			8	40	21	7	1	3					81
170			1	1	4	15	10	1	2							34
180					1	1	1	1								4
190		1[1]						1								2
	Total	1	2	4	15	46	120	141	128	94	68	16	12	3	2	652

[1]) $l = 20.4$.

smaller indices correspond with the short lengths. Table 3 contains few short lengths, mostly great and very great; the negative correlation is especially perceptible here.

In table 4 also we see that the average indices of the shortest length classes increase with the increase of the length classes. Only after the shortest length classes, so in the somewhat higher and high ones, do the average indices decrease with the increase of length classes.

Before we discuss the further significance of this grouping, it should be borne in mind that, with an investigation of the relation of L and LB-index, the spurious correlation must be taken into consideration. There is a great negative spurious correlation between L and LB[1]). For

[1]) Proc. Royal Neth. Acad. Amsterdam, **40**, 457 and 458, tab. 2 and tab. 3 (1937).

TABLE 3. Simplified correlation table 3. I-line in growth; 2nd (later) picking 1937. Correlation of L and LB. The length classes increase by 1 mm, the index classes by 2 units.

L	Ind.	48	50	52	54	56	58	60	62	64	66	68	70	72	74	Total
110										1						1
120										1		3	1			5
130								2	5		1	1				9
140							2			1	2	1	1			7
150							1		2	3	3	3		1		13
160						2	2	4	2	5	2	2		1		20
170						5	1	5	2	6	4	3	3			29
180					2	5	8	6	11	8	4	1				45
190		1		1	4	7	15	13	22	13	2	3	2			83
200				2	6	17	22	18	21	13	3					102
210		1	1	2	11	9	24	19	15	9						91
220				2	6	8	11	13	5	3	1					49
230			1			5	8	6	3							23
240				1			5	3	1							10
250							2	1								3
	Total	2	2	8	29	58	101	90	89	63	22	17	7	2		490

the material of our tables 1—3 we have calculated the spurious correlation. For this purpose we have regrouped the material, i.e. we have left the sequence in the length of the beans as it was, but beside the lengths we have placed the breadths in the reverse order and from these lengths and breadths we have calculated the indices. Proceeding along these lines, we find very low and very high indices. A very short length can occur with a very great breadth, then we find a very high index (we find values higher than 100 up to 300); a very great length can occur with a very narrow breath, then we find a very low index (the lowest index is 10). If we calculate the average indices belonging to the successive length classes, we find a very regular falling progression, indicating a great negative "spurious" correlation. This great negative spurious correlation is indeed found by calculation (Tab. 5).

The negative spurious correlation is great and the results for the 3 cases calculated (table 5) correspond very well. For the first 2 cases, i.e. for the material of the first picking of the I-line and the II-line, we find a very high standard variation for the LB-index; this was to be expected with the very great variation breadth in these cases. This very great

TABLE 4. The average *LB*-indices for increasing length classes of beans in their growth. 1937. I- and II-line.

I-line

Length in 0.1 mm	First picking N.	First picking *LB*-index $M \pm m$	First picking *LB*-index $\sigma \pm m$	Second picking N.	Second picking *LB*-index $M \pm m$	Second picking *LB*-index $\sigma \pm m$
— 30	3	60.2				
31— 40	9	58.5				
41— 50	25	60.7				
51— 60	26	61.5 ± 0.8	4.2 ± 0.6			
61— 70	49	63.9 ± 0.6	4.0 ± 0.4			
71— 80	50	66.3 ± 0.4	3.0 ± 0.3			
81— 90	39	66.2 ± 0.6	3.6 ± 0.4			
91—100	39	66.8 ± 0.4	2 5 ± 0.3			
101—110	34	66.1 ± 0.5	2.7 ± 0.3	1	65	
111—120	26	67 ± 0.4	2.2 ± 0.3	4	70.3	
121—130	27	66.1 ± 0.5	2.2 ± 0.3	9	63.6	
131—140	31	65.4 ± 0.4	2.3 ± 0.3	7	65.9	
141—150	42	64.2 ± 0.4	2.5 ± 0 3	13	67	
151—160	29	64 ± 0.5	2.5 ± 0.3	20	63.9	
161—170	53	63.6 ± 0.4	2.6 ± 0.2	29	64.2 ± 0.8	4.6 ± 0.6
171—180	36	64 ± 0.4	2.6 ± 0.3	45	62.3 ± 0.5	3.6 ± 0.4
181—190	35	63.1 ± 0.5	2.7 ± 0.3	83	62.05 ± 0.4	3.9 ± 0.3
191—200	20	62.7		102	61 ± 0.3	3.2 ± 0.2
201—210	25	61.5		91	59.8 ± 0.3	3.2 ± 0.2
211—220	8	63.1		49	59.9 ± 0.5	3.3 ± 0.3
221—230	1	64		23	59.7	
231—240				10	59.8	
241—250				3	60.3	
Total	607		Total	489		

II-line

Length in 0.1 mm	N.	*LB*-index $M \pm m$	*LB*-index $\sigma \pm m$	Length in 0.1 mm	N.	*LB*-index $M \pm m$	*LB*-index $\sigma \pm m$
— 30	5	69.9		Brot. Fwd.	242		
31— 35	5	68.8		111—120	54	73.7 ± 0.3	2.5 ± 0.2
36— 40	9	73.4		121—130	47	72 ± 0.4	2.7 ± 0.3
41— 50	31	73.8 ± 1.2	6.7 ± 0.9	131—140	88	72.8 ± 0.3	3.0 ± 0.2
51— 60	24	78.1		141—150	102	71.3 ± 0.3	3.1 ± 0.2
61— 70	40	77.4 ± 0.8	5.0 ± 0.6	151—160	81	70.1 ± 0 3	2.7 ± 0.2
71— 80	30	77.5 ± 0.5	2.5 ± 0.3	161—170	34	67.3 ± 1.3	2.5 ± 0.3
81— 90	24	76.5		171—180	4	67.1	
91—100	42	76.8 ± 0.4	2.4 ± 0.3	181—185	1	57.5	
101—110	32	75.8 ± 0.4	2.3 ± 0.3				
Carrd. Fwd.	242			Total	653		

TABLE 5. Correlation. Results.

r	n	$r \pm m$	$l \pm m$	$\sigma_l \pm m$	$LB \pm m$	$\sigma_{LB} \pm m$
Bruto correlation L and LB						
a. II-line; $l = 3.0 - 7.5$ mm	130	$+0.29 \pm 0.07$	5.46 ± 0.11	$1.28 \pm 0\ 08$	75.4 ± 0.57	6.45 ± 0.4
b. id. $l = 3.0 - 6.0$ „	76	$+0.28 \pm 0.1$	4.58 ± 0.1	0.9 ± 0.07	74.2 ± 0.8	7.15 ± 0.6
c. id. $l = 7.5 - 20.5$ „	538	-0.67 ± 0.024	12.9 ± 0.11	2.56 ± 0.08	72.2 ± 0.18	4.25 ± 0.13
a. I-line; $l = 3.0 - 8\ 5$ „	185	$+0.37 \pm 0.063$	6.24 ± 0.1	1.39 ± 0.07	63.7 ± 0.3	4.25 ± 0.2
b. id. $l = 8.5 - 23.0$ „	446	-0.46 ± 0.037	14.31 ± 0.18	3.78 ± 0.13	63.1 ± 0.16	3.4 ± 0.11
I-line 2nd pi. $l = 11.0 - 25.0$	489	-0.42 ± 0.017	19.09 ± 0.18	2.41 ± 0.08	61.1 ± 0.15	3.92 ± 0.12
L and Bd					$BD \pm m$	$\sigma_{BD} \pm m$
I-line: $l = 10.5 - 24.5$ mm	492	$+0.113 \pm 0.044$	$19. \pm 0.11$	2.42 ± 0.08	67.5 ± 0.3	6.35 ± 0.2
L and B			$L \pm m$		$B \pm m$	$\sigma_B \pm m$
a. I-line; $l = 3.0 - 8.5$ mm	180	$+0.96 \pm 0.006$	6.42 ± 0.1	1.38 ± 0.07	4.13 ± 0.08	1.06 ± 0.06
b. id. $l = 9.0 - 15.0$ „	243	$+0.95 \pm 0.006$	11.93 ± 0.13	2.01 ± 0.09	7.83 ± 0.08	1.3 ± 0.06
c. id. $l = 15.5 - 25.0$ „	669	$+0.84 \pm 0.011$	18.96 ± 0.08	1.98 ± 0.05	11.65 ± 0.05	1.06 ± 0.03
Spurious correl tion L and LB					$LB \pm m$	$\sigma_{LB} \pm m$
a. I-line; 1. pi. $l = 2.7 - 23$ mm	601	-0.73 ± 0.02	12.34 ± 0.2	4.93 ± 0.14	79.7 ± 0.23	57.4 ± 1.6
b. II-line; $l = 2.9 - 20.4$ „	659	-0.805 ± 0.014	11.54 ± 0.15	3.78 ± 0.10	82.9 ± 2.3	59.1 ± 1.6
c. I-line; 2. pi. $l = 11.3 - 24.6$ „	485	-0.76 ± 0.02	19.02 ± 0.11	2.41 ± 0.08	62.2 ± 0.5	12.03 ± 0.4

variability is the reason why the spurious correlations are rather divergent here. Of the formula $r = \frac{\Sigma pa_x a_y - nb_x b_y}{n\varsigma x \varsigma y}$ (JOHANNSEN 1926, p. 352) which we are using, the term $\Sigma pa_x a_y$, in which a_y represents the deviations of the average index, is very variable. With a still greater amount of material the results would have corresponded still more. The probable error of the correlation coefficient has been calculated by us according to the formula $m = \frac{1-r^2}{\sqrt{n}}$ (JOHANNSEN 1926, p. 355).

It is clear that we meet with spurious correlation in our calculations. We calculate the correlation of L and LB, so of L and $\frac{100\,B}{L}$. If L is arranged according to successive lengths or in successive length classes and the breadths are arbitrarily placed promiscuously, then with every L there comes a B which, in so far as its size is concerned, has the same

chance if only the number of breadths is sufficiently great. The value of the index is consequently regulated by the length: with an increasing length the index $\frac{100B}{L}$ will fall. If the lengths are joined to length classes, and the average breadth is to be calculated from greater numbers, then the course of the average indices with increasing length classes will be still more regular. It is clear that, if the lengths are arranged arbitrarily, as in our tables of spurious beans, the result is the same. There is then also with each length the same chance for the breadth as regards its size and thus with great lengths the smaller indices will correspond, which is proved if we combine lengths and indices into one table according to increasing lengths.

With our material of growing beans of the I-line of table 1, the arrangement was as follows. From the first, most plants, some pods with very small beans and a few with very large beans were picked. Added were the beans of 2 plants with small beans. In this manner the sequence of the lengths was taken for the calculation of the spurious correlation. The breadth was taken in the reverse order [1]). Thus, as opposed to very short lengths, we have sometimes very narrow breadths; the index will then be lower than 100. Generally as opposed to very short lengths we have great breadths, for there are many more of those in the material: the index in these cases is very high, higher than 100. In odd cases a very great breadth will be placed with a very short length and consequently we shall then meet with the highest indices, up to 300. Likewise, where the greatest lengths occur with the narrowest breadths, we shall find the lowest indices. If we have at our disposal a great number of observations, the whole series of breadth possibilities will agree with each length and more readily with each length class, i.e. the same average breadth will be found and with increasing length a falling average index. This we also see in our results.

The spurious correlation may be calculated by calculating the index of each spurious bean. The calculation has been made in this manner. The average breadth of the spurious beans may also be calculated for each length or for each length class and afterwards the index of the length and the average breadth may be determined.

As the variation breadth of the indices of the various classes is very great, the number of observations is not always sufficiently large to guarantee a perfectly regular fall of the average indices for increasing length classes with the beans of the 1st picking of the I-line. (tab. 6).

Thus the variation breadth of the LB-index of e.g. beans with a length of 5.0—6 mm extends from 55—235, etc. It must also be observed that the various length classes are not uniformly represented in the material. He who collected the material picked pods with small and with large

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1) FRETS, Proc. Royal Neth. Acad. Amsterdam, l.c. p. 6.

TABLE 6. Simplified correlation table. I-line in growth. 1937. First picking. Spurious correlation. Length classes = 1 mm; LB-index classes = 20 units.

Index-LB.

L	0	20	40	60	80	100	120	140	160	180	200	220	240	260	280	300	320	340	360 380	Total
30																1			2	3
40				1		2								3	2	1				9
50					3	3	4			1	4	4	4	1	1					25
60			1	4	4			1	2	3	7	4								26
70		3	6	9	4	1	1	3	13	7	2									49
80		1	11	8	5	5	4	9	7											50
90		1	6	5	3	6	12	6												39
100		6	4	3	7	11	7		1											39
110		1	4	3	11	11	3													33
120			1	9	11	5														26
130		1	6	16	1	3														27
140		4	12	8	7															31
150	1	9	17	8	7															42
160	1	8	14	5	2															30
170	12	16	20	2	1															51
180	6	19	8	3																36
190	5	18	9	3																35
200	2	12	6																	20
210		19	7																	26
220	2	4	2																	8
230		1	1																	2
Total	29	123	134	88	66	47	31	19	23	11	13	8	4	4	3	2			2	607

beans and only few pods with medium-size beans. This explains the gap, the hiatus, that exists in the variation breadth of the "spurious" beans of some length classes. The low length classes of 3.1—4.0 mm contain 3 "spurious" beans with low indices, 80, 110 and 115, and 6 "spurious" beans with very high indices of 270—303. (Tab. 6).

The low indices are of spurious beans where the short lengths (3.1—4.0 mm) occur together with narrow breadths and the very high indices are of beans where the same short lengths (3.1—4.0 mm) occur with very great breadth. As only few medium-size breadths occur in the material, the medium-size indices are not so much represented either.

We have made corresponding calculations for the material of line II and for the beans of line I of a later crop. The spurious correlation of L and $L\,B$ is in all three cases negative and great (tab. 5).

It is clear that spurious correlation must yield a fairly constant value. The spurious correlation of L and $L\,B$ is dependent upon L and B, but with a large amount of material the spurious correlations for different material will differ very little; they are determined by the absolute value of L and B and of their variability.

We have also determined the total correlation of L and $L\,B$ (tab. 5). As the correlation tables 1 and 2, for material of the I- and of the II-line respectively, clearly indicate a change of direction of the distribution of the indices over the table, 2 correlations have been calculated for each table, viz. one for the upper portion of the table, where the indices ascend from low to high, and a second for the lower portion of the table, where the indices descend from high to low.

For table 1 the correlation is thus determined of L and $L\,B$ of length classes from 3.0—8.5 mm, and from 8.5—23.0 mm, and for table 2 the correlation is determined of L and $L\,B$ of length classes from 3.0—7.5 mm (and 3.0—6.0) and from 7.5—20.5 mm.

In agreement with the direction of the course of the indices on the correlation table, we find for each table a positive correlation for the material of the shortest length classes and a negative correlation for those of the remaining length classes.

The negative total correlation of L and $L\,B$ is considerably smaller than the spurious correlation; consequently there is also an organic positive correlation of significant size.

In a second communication we shall still further discuss our results.